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Relaxation Methods Applied to Engineering Problems. XII. Fluid Motions Characterized by 'Free' Stream-Lines

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RELAXATION METHODS APPLIED TO ENGINEERING PROBLEMS

XII. FLUID MOTIONS CHARACTERIZED BY 'FREE' STREAM-LINES

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This paper deals with problems which are like the percolation problems of Part VII (Shaw & Southwell 1941) in that a double condition, imposed at a boundary initially unknown, replaces the more usual single condition at a specified boundary. They relate to 'free' stream-lines in the hydrodynamical theory of inviscid fluids.

For plane two-dimensional (steady) motions, the device of conformal transformation has led in the hands of Helmholtz, Kirchhoff and Rayleigh to a variety of solutions; but up to the present it has not taken account of gravity, and it would not seem capable of extension to motions characterized by axial symmetry. Relaxation Methods, in virtue of their tentative approach, here deal successfully with some problems hitherto unsolved.

Introduction

1. The use of conformal transformation to determine flow-patterns characterized by 'free' stream-lines is one of the major achievements of classical hydrodynamics. Writing

$$z = x+iy, \quad w = \phi+i\psi, \tag{1}$$

where ϕ is the velocity-potential and ψ the stream-function of the (two-dimensional steady) flow, we have

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} = u-iv, \tag{2}$$

u, v standing for the component velocities. Consequently, if

$$q^2 = u^2+v^2, \quad \theta = \tan^{-1}(v/u), \tag{3}$$

so that q measures the resultant velocity, θ its inclination to the axis of x, then

$$\zeta = \theta + i\log q = i\log(u-iv) = i\log\frac{dw}{dz}, \tag{4}$$

—i.e. ζ, as well as w, is a function of the complex variable z. If, then, the boundary can be specified both in the plane of z and of ζ, and if a formula is discoverable whereby either of the regions so delimited can be conformally transformed into the other, equation (4) may be expressed in the form

$$\frac{dw}{dz} = \exp(-i\zeta) = F(z), \tag{5}$$

and hence, by a single integration, w may be derived as a function of z. Now the real part of ζ has a known constant value on any part of the boundary which is straight, and *in the absence of gravitational forces* the imaginary part of ζ has a constant value, calculable from Bernoulli's equation, along any 'free' stream-line; so the way is clear to a solution of any problem in which every part of the boundary is either straight or 'free'. Examples have been treated by Helmholtz (1868), Kirchhoff (1869), Rayleigh (1876*b*) and others.

But the form in the ζ-plane is *not* known initially either of a fixed boundary which is curved or of a free stream-line when gravity is operative. As shown by von Karman (1940, §§ 6–7), one is then confronted (in an orthodox attack) with the difficulty that the boundary conditions are no longer linear. Levi-Civita (1907) devised transformations leading to solutions for a few curved boundaries, and his theory has been extended by other investigators; but allowance for gravity seems to have been made in one problem only, and then only approximately,—the problem of 'permanent' waves of finite amplitude. A less restricted treatment will have wide utility.

2. Some closely similar problems in the theory of percolation were shown in Part VII of this series (Shaw & Southwell 1941) to be tractable by Relaxation Methods: in view of that success, it seemed worth while to examine whether the methods can be used to deal with problems of the kind discussed above. This paper records some results of our examination. Here too the double boundary condition can be interpreted (in accordance with 'Prandtl's membrane analogue') as requiring a specified displacement to result from a loading of specified line-intensity, and here too—though the computational difficulties are greater—it has yielded to Relaxation Methods, thanks to their tentative approach.

Moreover when gravity effects are negligible the methods can be applied to problems in which the flow has axial symmetry, whereas the classical method, being dependent on the theory of a complex variable, is restricted to laminar motions. Here we have treated (i) a 'Borda mouthpiece' and (ii) an 'orifice plate' in a straight *circular* tube, and in (ii) we have been able to make allowance for gravity acting along the axis of the tube.

Finally, §§ 44–50 of the paper deal with 'free' stream-lines occurring in flow past fixed and rigid bodies of curved profile. Here, for the first time in this series, we have found solutions of a *type* which apparently is new.

Basic theory for laminar (two-dimensional) flow

3. The problem is to determine ϕ and ψ (§ 1) in a field of which the boundaries are not completely specified. Such parts as are specified represent fixed and rigid surfaces on which ψ has to take constant values because the normal component of velocity (measured by $\partial\psi/\partial s$) must vanish; the other parts are not known initially, being lines of constant ψ along which the pressure p as well is constant. Such lines are termed 'free' stream-lines. Their shapes are determined by a double boundary condition, so must emerge as a result of computation.

Assuming that the system is conservative, we have

$$\frac{p}{\rho}+\tfrac{1}{2}q^2+gy=\text{const.} \tag{i}$$

everywhere, by Bernoulli's theorem, y being measured upwards from some datum level; consequently (ρ being constant) at all points where p has a constant value p_0,

$$q^2+2gy=\text{const.}=2gy_0, \tag{ii}$$

y_0 defining the highest level to which such points can extend (i.e. the 'stagnation level' at which the velocity q is zero and the pressure is p_0). In (ii),

$$q^2=u^2+v^2=\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2. \tag{iii}$$

4. y being measured as above, x (as is customary) will be measured from left to right. If now we reverse the directions of x and y, measuring y downwards from the 'stagnation level' and x from right to left (figure 1), equation (ii) simplifies to

$$q^2 = 2gy, \tag{iv}$$

and the form of (iii) is unaltered. Along any stream-line ψ is constant, so q is measured by its normal gradient $\partial\psi/\partial\nu$: consequently on any 'free' stream-line we have the double boundary condition

$$\left.\begin{aligned} &\psi = \text{const.} \\ \text{and} \quad &\left(\frac{\partial\psi}{\partial\nu}\right)^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = q^2 = 2gy, \end{aligned}\right\} \tag{6}$$

according to (iii) and (iv).

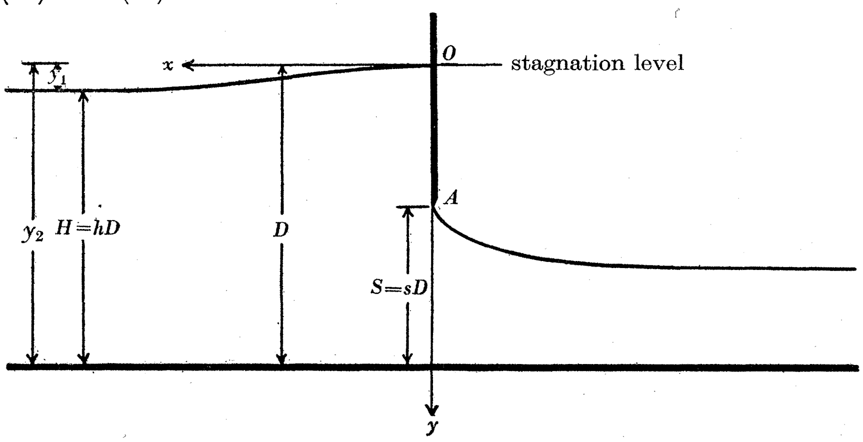

FIGURE 1

In some problems gravity is unimportant, i.e. g may be made zero in (ii). Then, in place of (6), we have on a free stream-line

$$\left.\begin{aligned} &\psi = \text{const.} \\ \text{and} \quad &\left(\frac{\partial\psi}{\partial\nu}\right)^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = q^2 = \text{const.} = k^2 \text{ (say)}, \end{aligned}\right\} \tag{7}$$

k^2 depending on the pressure and velocity of the fluid at entry; so again a double boundary condition has to be satisfied.

In either class of problem, everywhere, ψ satisfies the equation

$$\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{8}$$

Basic theory for flow possessing axial symmetry

5. Flow characterized by axial symmetry is discussed by Lamb (1932, § 94), whose treatment we shall follow with some minor changes of notation. Oz being the axis of symmetry, the total flow across a circular area of radius r is denoted by $2\pi\psi$: then, when r is invariant,

$$\begin{aligned} -2\pi\delta\psi &= \textit{outward} \text{ radial flow through an annulus } 2\pi r\delta z, \\ &= u \,.\, 2\pi r\delta z, \text{ say, } u \text{ denoting the radial component of velocity;} \end{aligned}$$

when z is invariant,

$$2\pi\delta\psi = \text{flow } \textit{from left to right} \text{ through an annulus } 2\pi r\delta r,$$
$$= v\,.\,2\pi r\delta r, \text{ say, } v \text{ denoting the axial component of velocity.}$$

Accordingly
$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad v = \frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{9}$$

and so, if the flow is irrotational, the governing equation is

$$0 = r\left(\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z}\right) = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}. \tag{10}$$

On any stream-line ψ is constant as before, but now the resultant velocity $q = \frac{1}{r}\frac{\partial\psi}{\partial \nu}$. Consequently on a 'free stream-line' we have the double condition

$$\left.\begin{aligned} &\psi = \text{const.} \\ \text{and} \quad &\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\nu}\right)^2 = \frac{1}{r^2}\left\{\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right\} = q^2 = \text{const.} = k^2 \text{ (say)}\end{aligned}\right\} \tag{11}$$

when gravity is inoperative, or

$$\left.\begin{aligned} &\psi = \text{const.} \\ \text{and} \quad &\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\nu}\right)^2 = q^2 = 2gz + \text{const.}\end{aligned}\right\} \tag{12}$$

when gravity acts in the axial direction Oz.

Implications of the double boundary condition

6. The constant in the first of (6), (7), (11) or (12) is equal or proportional to the total rate of flow, therefore (normally) will not be predictable but must emerge as a result of computation. For example, by attaching a value of k^2 in (7) or (11) we specify the total pressure drop, which is one of the factors on which the rate of flow depends; but it also depends on the resistance offered by the fixed surfaces, and this (initially) is unknown. Equally the resistance is unknown in problems where gravity is operative, e.g. when the double boundary condition is expressed by (6): then, having specified the stagnation point (i.e. the energy of the incoming stream), we again have no initial knowledge of the total rate of flow.

Alternatively the rate of flow may be specified, i.e. a value may be imposed on ψ for the free stream-line; but then the pressure drop is not predictable, so we shall not know what value to attach to k in (7) or (11), or the stagnation level when working with equations (6). Only exceptionally (e.g. for the 'Borda mouthpiece' treated in §§ 18–22) is it possible completely to formulate the computational problem in advance: normally one must proceed by trial and error, giving different values in turn to some parameter,* and interpolating to obtain the wanted solution. The necessity will appear in our treatment of particular examples.

* Occasionally it will be desirable to vary some dimension of the *rigid surfaces*. (Cf. the treatment of the 'drowned sluice' in §§ 38 and 39.)

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Reduction of the equations to non-dimensional form

7. Before we can proceed to computation, 'dimensional' factors must be eliminated from the equations. Let D be some representative dimension of the rigid boundaries, and in relation to § 4 write

$$x = x'D, \quad y = y'D, \quad \nu = \nu'D, \quad \psi = \psi'\sqrt{(2gD^3)}. \tag{13}$$

Then (for a free stream-line on which the pressure is p_0) we have, in place of (6),

$$\left.\begin{aligned} &\psi' = \text{const.} \\ \text{and} \quad &\left(\frac{\partial\psi'}{\partial\nu'}\right)^2 = \left(\frac{\partial\psi'}{\partial x'}\right)^2 + \left(\frac{\partial\psi'}{\partial y'}\right)^2 = q'^2 = y', \end{aligned}\right\} \tag{i}$$

and similarly, in place of (7),

$$\left.\begin{aligned} &\psi' = \text{const.} \\ \text{and} \quad &\left(\frac{\partial\psi'}{\partial\nu'}\right)^2 = \left(\frac{\partial\psi'}{\partial x'}\right)^2 + \left(\frac{\partial\psi'}{\partial y'}\right)^2 = q'^2 = \text{const.} = k'^2, \end{aligned}\right\} \tag{ii}$$

where $k^2 = 2gDk'^2$. The form of (8) is conserved, ψ, x, y, ν being replaced by ψ', x', y', ν'.

In relation to § 5 write

$$r = r'D, \quad z = z'D, \quad \nu = \nu'D, \quad \psi = \psi'\sqrt{(2gD^5)}. \tag{14}$$

Then the form of (10) is conserved, ψ, r, z, ν being replaced by ψ', r', z', ν'; in place of (11) we have

$$\left.\begin{aligned} &\psi' = \text{const.} \\ \text{and} \quad &\frac{1}{r'^2}\left(\frac{\partial\psi'}{\partial\nu'}\right)^2 = \frac{1}{r'^2}\left\{\left(\frac{\partial\psi'}{\partial r'}\right)^2 + \left(\frac{\partial\psi'}{\partial z'}\right)^2\right\} = q'^2 = \text{const.} = k'^2, \end{aligned}\right\} \tag{iii}$$

where $k^2 = 2gDk'^2$ as before; and in place of (12)

$$\left.\begin{aligned} &\psi' = \text{const.} \\ \text{and} \quad &\frac{1}{r'^2}\left(\frac{\partial\psi'}{\partial\nu'}\right)^2 = q'^2 = z' + \text{const.} \end{aligned}\right\} \tag{iv}$$

Hereafter we shall suppress the dashes attached to x, y, r, z, ν, q, ψ, which accordingly will carry 'non-dimensional' significance. On that understanding, in § 4, (7) and (8) still hold but (6) is replaced by

$$\left.\begin{aligned} &\psi = \text{const.} \\ \text{and} \quad &\left(\frac{\partial\psi}{\partial\nu}\right)^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = q^2 = y; \end{aligned}\right\} \tag{15}$$

in § 5, (10) and (11) still hold but (12) is replaced by

$$\left.\begin{aligned} &\psi = \text{const.} \\ \text{and} \quad &\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\nu}\right)^2 = \frac{1}{r^2}\left\{\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right\} = q^2 = z + \text{const.} \end{aligned}\right\} \tag{16}$$

Principles of the relaxational attack. The question of 'stability'

8. Our problems are like those discussed in Part VII (Shaw & Southwell 1941), in that a double boundary condition is imposed at a boundary initially unknown. As in Parts VIII and XI,* ψ being interpreted as the transverse displacement of a tensioned membrane, its normal gradient is to be interpreted as a line-intensity of transverse edge-loading: consequently the aim now is to find a form of boundary such that on it a constant displacement is maintained by loading operative only at that boundary, and on it having specified intensity; and for that purpose we may again adopt the notion (used in Part VII) of movable weights or 'shot bags.' What now makes the problem different is that whereas in Part VII the loading had uniform line-intensity with respect to a fixed direction, here its line-intensity (*a*) is specified with respect to the boundary perimeter s and (*b*) depends, when gravity is operative, on the y-coordinate of the point in question. On both accounts the mechanical (loaded membrane) system may be unstable, and solutions tend to diverge.

As regards (*a*), it is evident that any lengthening of s will entail a corresponding increase of the total edge-loading, so tentative modification of an assumed 'free stream-line' may lead (incorrectly) to a shape exhibiting local 'corrugations'. The effect of (*b*) is less evident, and must be discussed in relation to some particular example.

9. In this discussion nothing is gained by an elimination of 'dimensional' factors. Consider the case of flow along a flat bed and under a 'drowned sluice' (figure 1), and assume that both far upstream and far downstream of the sluice the flow is uniform and horizontal, so that

$$\psi = -Uy \tag{17}$$

when the bed is parallel with the axis of x, and when the flow is from left to right. Let $y = y_1$ at this 'asymptotic level' of the free stream-line, $y = y_2$ on the rigid bed.

Then the total rate of flow through the sluice is

$$Q = U(y_2 - y_1), \tag{i}$$

and the condition (6) requires that

$$U^2 = \left(\frac{\partial\psi}{\partial\nu}\right)^2 \text{ for the free stream-line,}$$

$$= 2gy_1. \tag{ii}$$

Eliminating U between (i) and (ii), we have

$$2gy_1 = \frac{Q^2}{(y_2 - y_1)^2}, \tag{18}$$

from which y_1 may be determined when Q and y_2 are known. It is a cubic having at most two real roots in the range $0 \leqslant y_1 \leqslant y_2$, as may be seen (figure 2) by plotting against y_1 (*a*) a straight line representing $2gy_1/Q^2$ and (*b*) a curve representing $(y_2 - y_1)^{-2}$.

10. Two depths of parallel stream can thus consist with specified values of Q and of y_2. Suppose first that the aim is to determine conditions corresponding with A, and that y_1 is underestimated. Then, in the membrane analogue, the loading $\sqrt{(2gy_1)}$ and the normal gradient $Q/(y_2 - y_1)$ will both be underestimated; but figure 2 shows that *the loading will be underestimated by more than the normal gradient* (if Op represents the assumed value of y_1, then

* Southwell & Vaisey (1943), Allen, Southwell & Vaisey (1944).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$pq < pr$), so the tension operating on the normal gradient will overcome the edge-loading, and the membrane will indicate a still lower value of y_1. If on the other hand y_1 is over-estimated, a similar argument shows that the loading will overpower the net tension so that a still higher value of y_1 will be indicated. Thus the mechanical system is *unstable*.

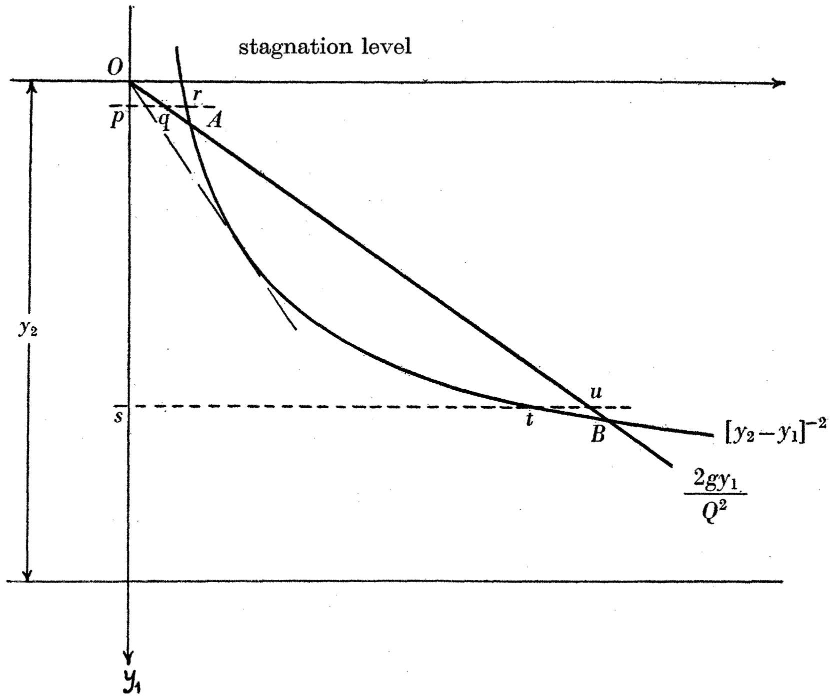

FIGURE 2

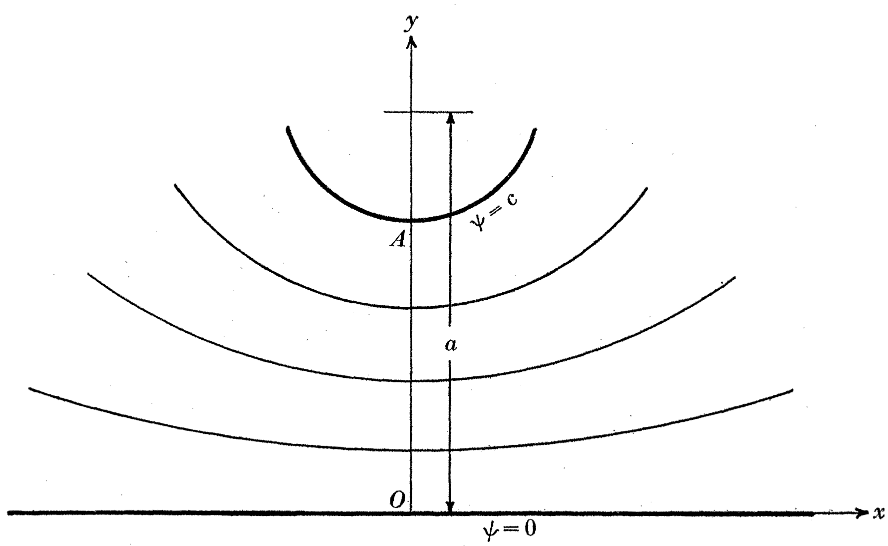

FIGURE 3

But it is stable when the aim is to determine conditions corresponding with B (figure 2), because then, if y_1 is underestimated (say, by assuming a value Os), the loading will over-power the net tension ($su > st$) and indicate a greater value for y_1.

It may be deduced from (18) or from figure 2 that Q regarded as a function of y_1 has its maximum value when $3y_1 = y_2$; also that as $Q \to 0$ one possible value of y_1 tends to 0 (giving a zero value to U), the other to y_2. When no account has to be taken of gravity—i.e. when (7) replaces (6)—the straight line through O, in figure 2, is replaced by a line parallel to Oy_1. Then (i) only one real value for y_1 exists, and (ii) the circumstances are like those which correspond with B—i.e. the mechanical system is *stable*.

11. §§ 9–10 relate to parallel streams in which

$$\psi = -Uy, \qquad (17)\ bis$$

simply, U being constant. Quite apart from the tendency of a free stream-line to 'corrugate' (§ 8), stability (in the membrane analogue) may give place to instability when the boundary is curved instead of straight.

Consider, for example, the plane-harmonic stream-function

$$\psi = \tfrac{1}{2}\mu \log \frac{x^2+(a+y)^2}{x^2+(a-y)^2} \quad (\mu \text{ constant}), \qquad \text{(i)}$$

$$= \mu \log \frac{a+y}{a-y} \quad \text{when } x = 0. \qquad \text{(ii)}$$

The stream-lines are coaxial circles including the axis $y = 0$ (for which $\psi = 0$), as shown in figure 3: any one of them may be made the boundary of a 'free' stream. Near the point $(0, a)$ they are closely packed, and at that point $\psi = \infty$.

At points on Oy the velocity (parallel to Ox) is given by

$$q = \frac{\partial \psi}{\partial y} = \frac{2\mu a}{a^2-y^2}, \qquad \text{(iii)}$$

and on the free stream-line ($\psi = c$), according to (ii),

$$y = a \tanh \frac{c}{2\mu} = y_c \text{ (say)}, \qquad \text{(iv)}$$

therefore according to (iii)

$$q = \frac{2c}{a} \Big/ \left(1-\frac{y_c^2}{a^2}\right) \log \frac{a+y_c}{a-y_c} = q_c \text{ (say)}. \qquad \text{(v)}$$

If now, keeping c constant, by an altered choice of boundary we decrease the breadth y_c of the stream at its narrowest section (OA, figure 3), according to (v) q_c will be increased only if

$$\frac{y_c}{a} \log \frac{a+y_c}{a-y_c} < 1, \qquad \text{(vi)}$$

that is, only if y_c/a has a value less than some fraction of unity. When $(1-y_c/a)$ is positive and small (i.e. when the boundary has sharp convex curvature), decrease of y_c entails a *decrease* of q_c.

This example is merely illustrative, for ψ as given by (i) does not satisfy either of the double boundary conditions (15) or (7); but it will serve to make the point that curvature of a free stream-line, *when of a sense which makes the fluid stream concave*, tends to make the membrane system unstable, and may neutralize its stability as deduced in §§ 9–10. One can, moreover, come to the same conclusion in respect of axially symmetrical flow. It has importance when, on account of gravity, wave motion makes an appearance: near the crest of a wave, stability of the membrane system may be expected; near a 'trough', either stability or instability according to the depth (a small depth making for stability).

12. In this last connexion an important theorem can be stated, due to Mr D. N. de G. Allen. It has been shown (§§ 9–10) that two depths of parallel stream (normally) can consist with specified values of the total flow (Q) and of its total energy (i.e. of y_2). h denoting the depth, so that $h = y_2 - y_1$ in (18), this means that the equation

$$X = Q^2 - 2gh^2(y_2 - h) = 0 \tag{19}$$

has two roots (normally) in the range $0 \leqslant h \leqslant y_2$. Denoting these by h_1 and h_2 ($h_2 > h_1$), we have

$$\left.\begin{array}{lll} X>0 & \text{when} & 0\leqslant h<h_1, \\ X<0 & \text{when} & h_1<h<h_2, \\ X>0 & \text{when} & h_2<h\leqslant y_2. \end{array}\right\} \tag{i}$$

Now on the free stream-line, by the second of (6),

$$\frac{\partial\psi}{\partial\nu} = \surd(2gy) = \surd\{2g(y_2-h)\}, \tag{ii}$$

and when μ is defined by
$$\frac{\partial\psi}{\partial\nu} = \mu Q/h \tag{iii}$$
the condition (ii) may be written as

$$\mu^2Q^2 - 2gh^2(y_2-h) = 0. \tag{iv}$$

Then from (19) combined with (iv) it follows that

$$\mu^2 = 1 - X/Q^2,$$

so (μ being positive) the inequalities (i) may be written as

$$\left.\begin{array}{lll} \mu<1 & \text{when} & 0\leqslant h<h_1, \\ \mu>1 & \text{when} & h_1<h<h_2, \\ \mu<1 & \text{when} & h_2<h\leqslant\ \ . \end{array}\right\} \tag{v}$$

Suppose that the free stream-line exhibits a point P of maximum depression (i.e. a 'trough'). At such a point $\partial^2\psi/\partial x^2 < 0$, and when the 'bed' is straight and horizontal it may be concluded that $\partial^2\psi/\partial x^2 \leqslant 0$, therefore $\partial^2\psi/\partial y^2 \geqslant 0$, at all points in the vertical through P. But on that understanding $(\partial\psi/\partial\nu)_P > Q/h$, so μ as defined in (iii) is greater than unity: therefore according to (v)

$$h_1 \leqslant h \leqslant h_2 \text{ at any 'trough'}. \tag{20}$$

Thus the theorem is established: *The free stream-line can nowhere fall below the lower of the two 'asymptotic levels' which were discussed in* §§ 9–10.

13. It can be seen from this discussion that the membrane analogue, applied to problems governed by (15), may sometimes entail a system not entirely stable; but it should not be concluded that it will have no value. Unstable systems have been confronted in earlier papers of this series, and their difficulties have been surmounted by special devices.

Now that the methods of Parts VIII and XI* are available, two obvious alternatives are presented: either (A) we may make ψ constant on the assumed free stream-line, compute ψ everywhere, and deduce values of the boundary gradient $\partial\psi/\partial\nu$; or (B) we may impose the specified boundary gradient on an assumed free stream-line, compute ψ everywhere, and deduce a shape of stream-line which can then be used as a new starting assumption. Either method may have advantages in relation to a particular example.

* Southwell & Vaisey (1943), Allen, Southwell & Vaisey (1944).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Details of the relaxational attack

14. The relaxational treatment of governing equations like (8) or (10) has been fully explained in earlier papers. First, the exact differential equation is replaced by an approximation in finite differences. Then, from this approximation, one or more 'relaxation patterns' are deduced. Finally, by a use of such 'patterns' we liquidate the residuals entailed by a more or less arbitrary starting assumption.

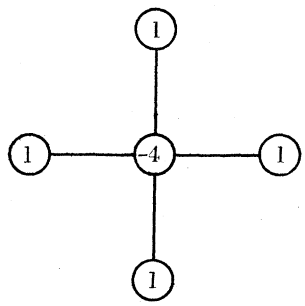

FIGURE 4

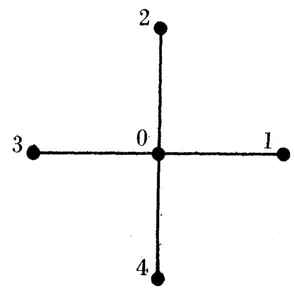

FIGURE 5

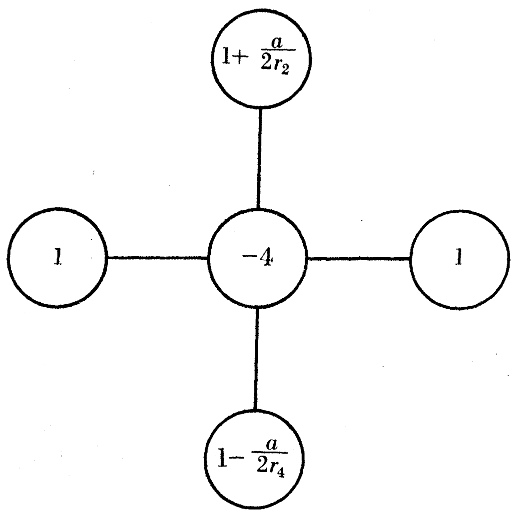

FIGURE 6

In laminar flow (§ 4) ψ must satisfy the plane-harmonic equation (8), of which (for a square net) the finite-difference approximation is

$$\mathbf{F}_0 = \Sigma_{a,4}(\psi) - 4\psi_0 = 0 \tag{21}$$

(cf. Part III,* § 8); the 'pattern' is as shown in figure 4. In flow possessing axial symmetry ψ is subject to (10), § 5, to which, accordingly, a like approximation is required. Suffixes 0, 1, 2, 3, 4 relating to the points so numbered in figure 5, and a (as usual) standing for a mesh-side of the chosen square net, we have, approximately,

$$2a\left(\frac{\partial\psi}{\partial r}\right)_0 = \psi_2 - \psi_4, \tag{i}$$

* Christopherson & Southwell (1938).

and (for $N = 4$, from (9) of Part III, § 8)

$$a^2\left(\frac{\partial^2\psi}{\partial r^2}+\frac{\partial^2\psi}{\partial z^2}\right)_0 = \psi_1+\psi_2+\psi_3+\psi_4-4\psi_0. \tag{ii}$$

Consequently the finite-difference approximation to (10) is

$$\mathbf{F}_0 = \psi_1+\psi_2+\psi_3+\psi_4-4\psi_0-\frac{a}{2r}(\psi_2-\psi_4) = 0, \tag{22}$$

and the pattern is as shown in figure 6.

In either type of system, if the free stream-line curves sharply as it leaves the rigid boundary, 'graded' nets can be used with advantage. Then, some strings will be inclined at 45° to the axes of x and y or of z and r; (21) will hold as before, but modifications must be made in (22). The point will arise in our examples (cf. § 24).

15. It is not necessary again to describe the liquidation process, or the treatment of irregular stars. But in regard to Method (A), § 13, it should be remarked that estimation of normal gradients is here made easier by the circumstance that the boundary is a stream-line. Since

$$\frac{\partial}{\partial \nu} \equiv \cos(x,\nu)\frac{\partial}{\partial x}+\sin(x,\nu)\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial s} \equiv \cos(x,\nu)\frac{\partial}{\partial y}-\sin(x,\nu)\frac{\partial}{\partial x},$$

when $\partial\psi/\partial s \equiv 0$, then
$$\frac{\partial\psi}{\partial x} = \cos(x,\nu)\frac{\partial\psi}{\partial \nu}, \quad \frac{\partial\psi}{\partial y} = \sin(x,\nu)\frac{\partial\psi}{\partial \nu}. \tag{23}$$

In laminar flow, $\partial\psi/\partial\nu$ must have a value given by the second of (7) or (15); consequently (23) can be used to determine the gradient which ψ should have, at the boundary, *in the direction of any string which cuts that boundary*. In axially-symmetrical flow, z and r replace x and y in (23), and $\partial\psi/\partial\nu$ must have a value given by the second of (11) or (16); so again we can decide the gradient which ψ should have at the boundary, using one or other of the formulae

$$\frac{\partial\psi}{\partial z} = \cos(z,\nu)\frac{\partial\psi}{\partial \nu}, \quad \frac{\partial\psi}{\partial r} = \sin(z,\nu)\frac{\partial\psi}{\partial \nu}. \tag{24}$$

Obvious modifications are needed when (§ 14) a string is inclined at 45° to the axes of x and y or z and r.

16. In Method (A), § 13, starting with an assumed form for the free stream-line we make ψ plane-harmonic in the fluid 'field', then deduce the slopes of the strings at points where they cut the assumed stream-line, and compare these with what they should be according to (23) or (24). Usually there will be discrepancies, indicating that the assumed form must be modified.

The nature of the required modification will depend on whether the membrane system (§§ 9–11) is stable or unstable, and sometimes this will not be known initially; but one or two trials will usually suffice to decide both the sense and the magnitude of the required movement. In this connexion the arbitrary nature of Method (A) may be an advantage.

17. Method (B) has no arbitrary feature of this kind,—applied to any assumed shape of boundary it yields a definite indication regarding the shape to be adopted in the next stage of the computations. If the membrane system were always stable so that successive shapes must converge, this could be deemed an advantage; but usually the shape is found to

diverge in some part at least, and in Method (B) such divergence is not under control. Even so the method may sometimes have value at the start of an investigation, as a means of deciding the trend of the wanted stream-line, and the regions in which it has a tendency to diverge, before its final detail is examined by Method (A). It must, however, be employed with caution.

Section I. Jets uninfluenced by gravity

Example 1. *Borda mouthpiece in laminar flow*

18. This use of Methods (A) and (B) in combination may be exemplified in relation to the problem shown in figure 7. Fluid passing down a channel AA' develops a free surface at the edges of a two-dimensional 'Borda mouthpiece' BB',—the whole flow being symmetrical with respect to a centre-line CC. The (uniform) velocity far upstream of BB' is specified, also the velocity (determined by the pressure) at the boundary of the issuing jet. We have to determine the stream-function ψ, and hence the form of the free stream-line BD, given that $\psi = 0$ at all points on CC, $\psi =$ const. at all points on $AEFBD$.

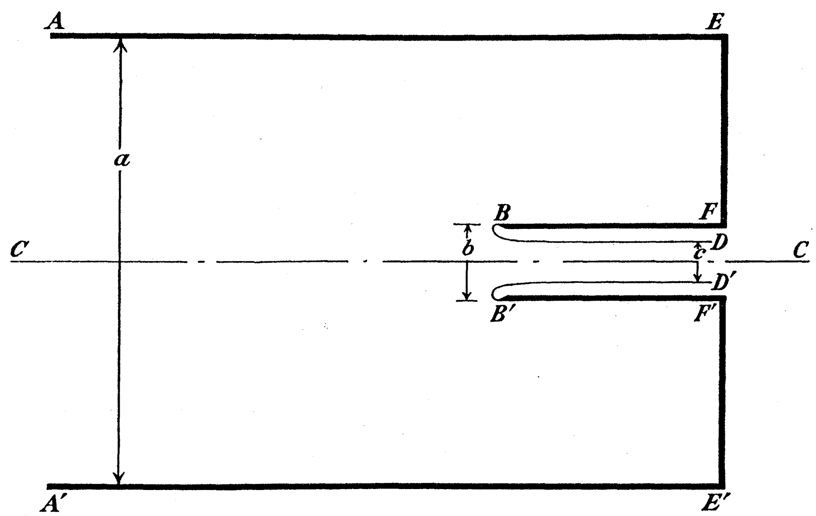

Figure 7

19. This is one of the exceptional problems (cf. § 6) which can be formulated completely at the outset. Let c denote the 'asymptotic breadth' of the free jet far downstream of BB', b the dimension BB' and a the dimension AA' in figure 7; and assume that the fluid in contact with the closed end (EF, $F'E'$) is sensibly at rest. Then, suppressing the gravity term in (i) of § 3, and writing

q_a, p_a for the uniform velocity and pressure far upstream,

0, p_0 for the uniform velocity and pressure at EF, $F'E'$,

q, p for the uniform velocity and pressure on BD, $B'D'$,

we have by Bernoulli's theorem

$$\frac{p_a}{\rho}+\tfrac{1}{2}q_a^2=\frac{p_0}{\rho}=\frac{p}{\rho}+\tfrac{1}{2}q^2, \tag{i}$$

and for conservation of momentum (ρ being invariant)

$$ap_a - (a-b)\,p_0 - bp = \text{rate of generation of momentum}$$
$$= \rho(q\,.\,cq - q_a\,.\,aq_a). \qquad \text{(ii)}$$

Moreover $$aq_a = cq, \qquad \text{(iii)}$$

because the inflow and outflow are identical.

Now eliminating p_a, p_0, p between (i) and (ii), we have

$$(b-2c)\,q^2 + aq_a^2 = 0,$$

whence, and from (iii), it follows that

$$a(b-2c)+c^2 = 0. \qquad (25)$$

When $a/c = \infty$, then $b = 2c$ according to (25), so the coefficient of contraction $c/b = 0{\cdot}5$ (a known result). Figure 8 shows the variation of c/b with a/b.

Given a and b in figure 7, we know c from (25).* Then (ψ being zero on CC) along $AEFBD$

$$\psi = \tfrac{1}{2}aq_a = \tfrac{1}{2}cq, \quad \text{by (iii)},$$
$$= \tfrac{1}{2}ck \qquad (26)$$

according to the second of (7), § 4. Thus the two constants in (7) can be related.

20. The exact solution is known for the limiting case in which a, and the length BF of the mouthpiece (figure 7), have infinite values. This case might be tractable by relaxation methods if the device of 'inversion' were employed to limit the field of computation; but a restricted field has greater practical importance.

Accordingly as a first example we treated the problem shown in figure 7, taking $a/b = 6$ so that (cf. figure 8)

$$c/b = 0{\cdot}5228$$

according to (25). We gave ψ the values 0 and 6, respectively, on the centre-line and on the rigid boundary $AEFB$, and a and b, consistently, the values 12 and 2; so the constant in the first of (7) was 6, and ck had the value 12 according to (26). Consequently k in the second of (7) had to be given the value

$$k = \frac{12}{c} = \frac{12}{b}\left(\frac{b}{c}\right) = \frac{6}{0{\cdot}5228} = 11{\cdot}4772, \qquad (27)$$

and thus the computational problem was completely formulated in advance. Actually, to eliminate decimals, we computed values of 100ψ or of 1000ψ.†

21. To test the value of Method (B) as a means of preliminary exploration (§ 17), we started from an assumption known to be incorrect—that from B the free stream-line drops immediately to its 'asymptotic distance' ($\tfrac{1}{2}c$) from the centre-line, turning sharply both at B and G as shown by 1 in figure 9.

* Only one solution is acceptable, since $c < b < a$.

† 100ψ in our application of Method (B) and in the coarser parts of the graded nets used with Method (A).

The anticipations of § 17 were realized: the indicated forms of the free stream-line were plausible in the earlier stages of computation, but in later stages they diverged from the accepted form (shown by a broken line in figure 9), and ultimately they tended to 'corrugate' (§ 8) near B.

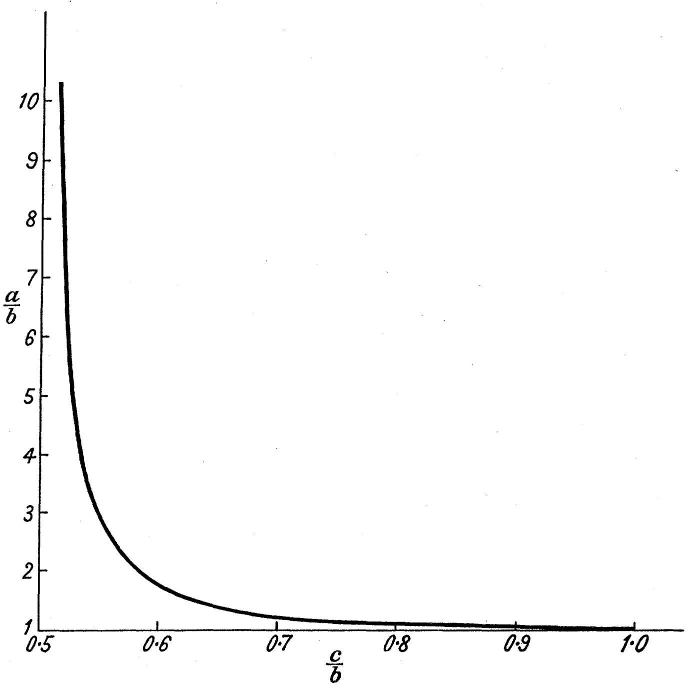

FIGURE 8

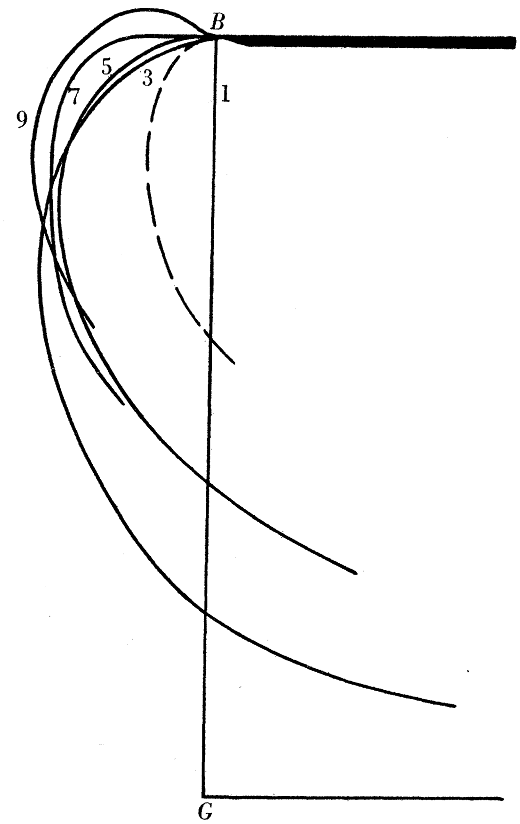

FIGURE 9

22. Accordingly the investigation was continued with a use of Method (A) and suitably 'graded' nets. Figures 10–12 present our accepted solution, in which the second of (7), viz.

$$\frac{\partial\psi}{\partial\nu} = k = 11{\cdot}4772, \quad \text{according to (27)}, \tag{28}$$

is satisfied within 0·7 % at all test points on the free stream-line. Accepted values of the boundary gradient (multiplied by 10) are recorded by numerals just outside the free stream.

The range of the grading in these diagrams is very wide. Thus in figure 10 the mesh-side a is $\frac{1}{2}$ in the regions remote from BB' in figure 7, but is graded down to $\frac{1}{16}$ in a region $BabcdefB$ which, to a larger scale, is reproduced in figure 11; and here again the net is graded, so that $a = \frac{1}{64}$ in the immediate neighbourhood of B. Figure 12 exhibits the accepted values of 1000ψ (and contours of 100ψ) in this finest part of the net. The requisite gradient (28) is obtained at points extending to within one mesh-length of B: nearer to B the free stream-line has not (strictly) been determined, so is shown by a broken line. This meets the rigid boundary at a point somewhat different from what was postulated; but the error (of the order of $\frac{1}{2}a = \frac{1}{128}$) is for practical purposes negligible.

For comparison, figure 11 also exhibits (in broken line) the stream-line which is given by

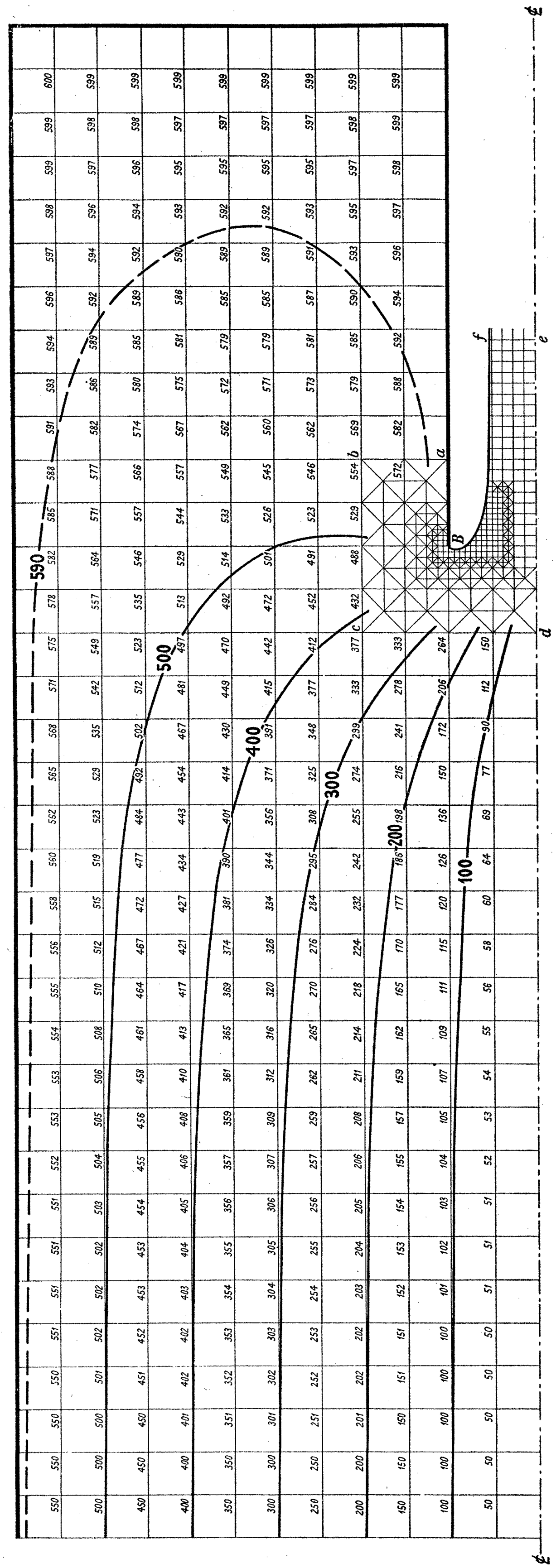

FIGURE 10. Values and contours of 100ψ.

orthodox analysis when outside the mouthpiece the fluid extends to infinity.* The difference is appreciable and easy to explain:—In the exact solution, fluid approaches B from all directions; but in our solution it is nearly stationary in the closed end which (figure 10) lies to the right of B: consequently the free stream-line would be expected to turn more rapidly, at B, towards the right, and this is what was found.

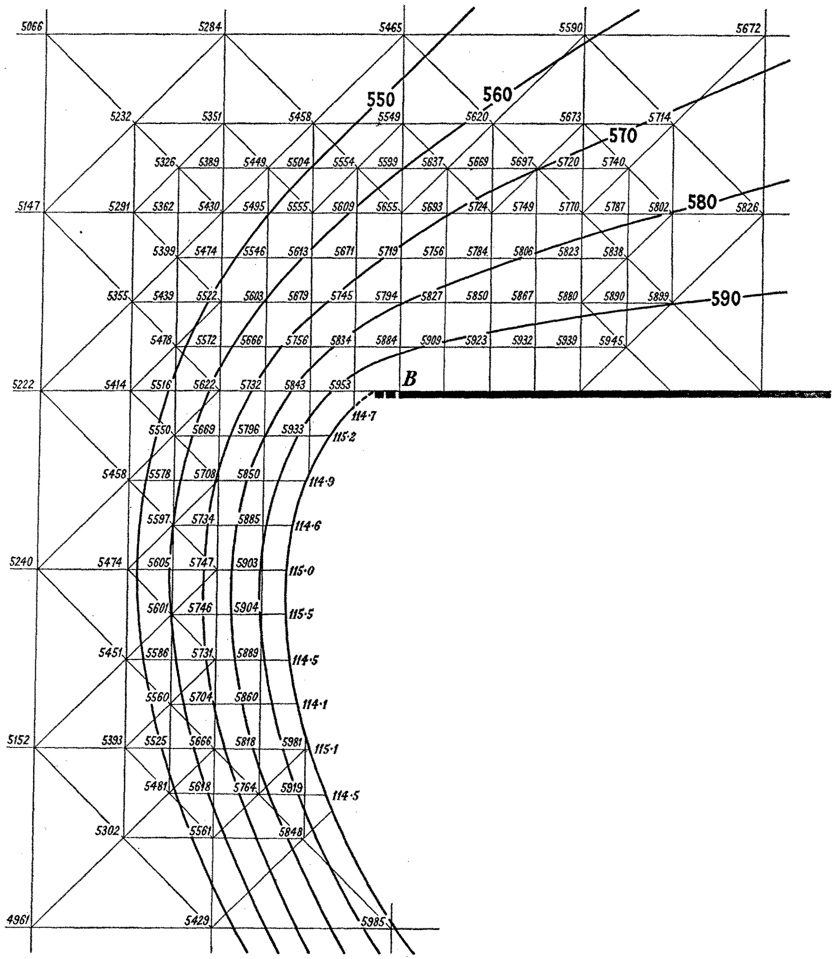

FIGURE 12. Values of 1000ψ and contours of 100ψ.

Example 2. Borda mouthpiece in axially symmetrical flow

23. In our second example, figure 7 was taken to represent the cross-section of a system having axial symmetry about CC, and calculations were made in accordance with § 5. The argument of § 19 again serves to determine the 'asymptotic breadth', if in (25) a, b and c (which there stand for relative areas of cross-section) are now replaced by a^2, b^2 and c^2 (which have a like significance). Then

$$a^2(b^2-2c^2)+c^4 = 0,$$

* Cf. Lamb, 1932, § 74. With origin at B, directions of x and y as shown in figure 1, and b and c defined as in figure 7,

$$x = \frac{b-c}{\pi}(\sin^2 \tfrac{1}{2}\theta - \log \sec \tfrac{1}{2}\theta), \quad y = \frac{b-c}{2\pi}(\theta - \sin\theta).$$

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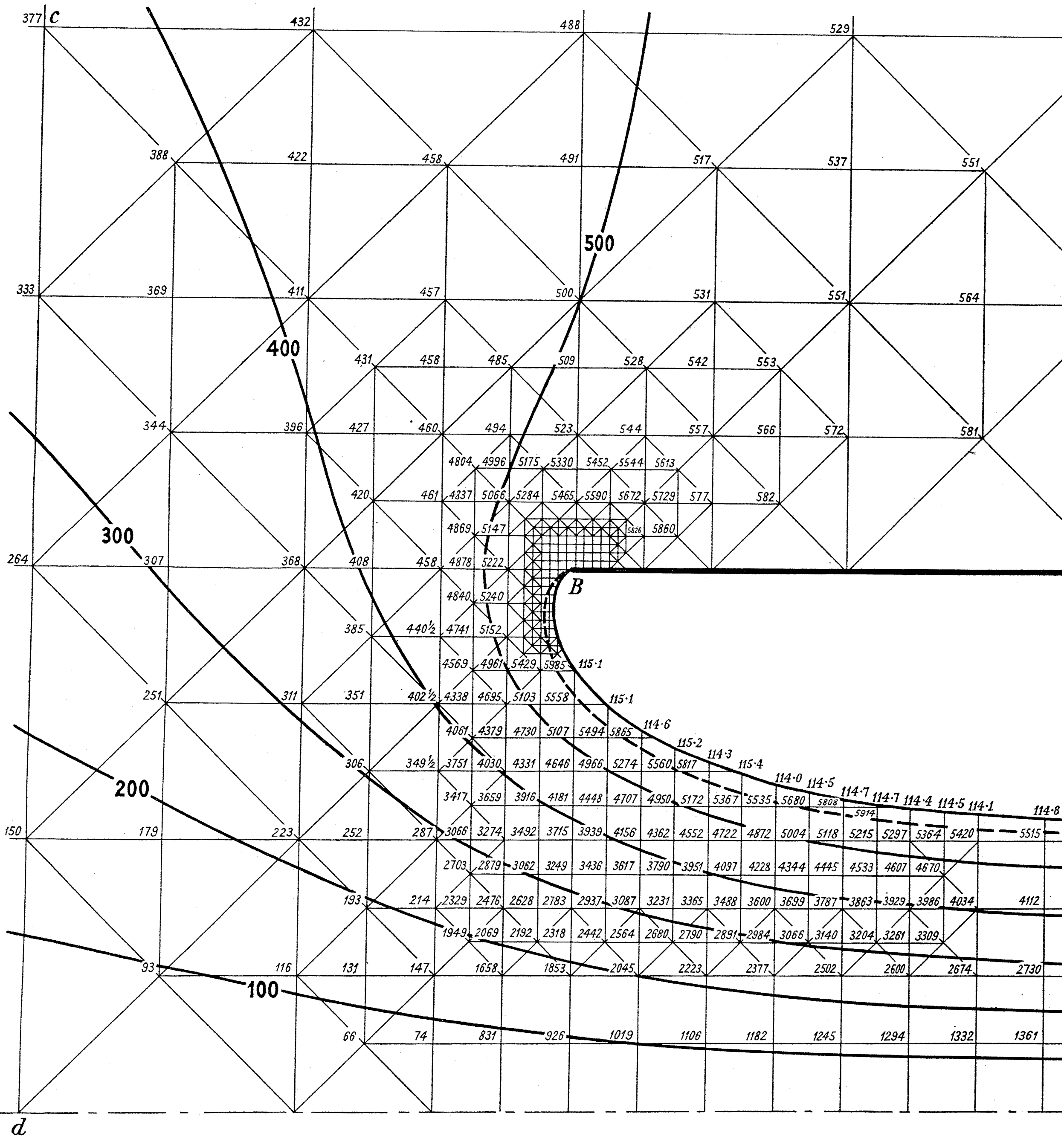
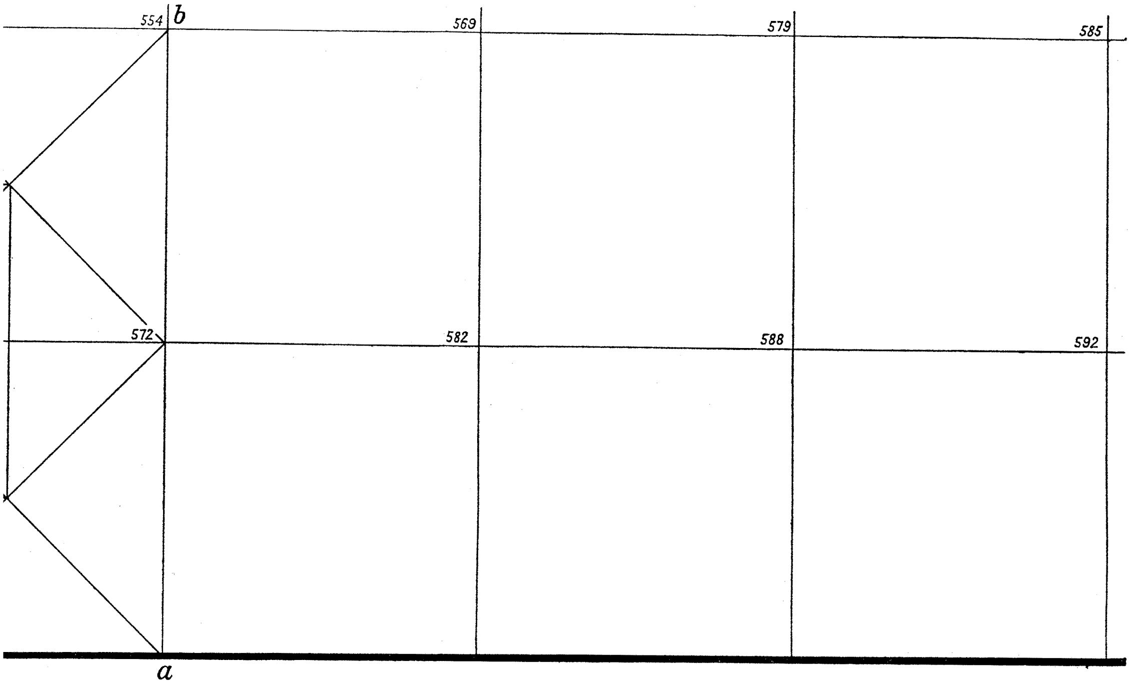


FIGURE 11. Values and contours



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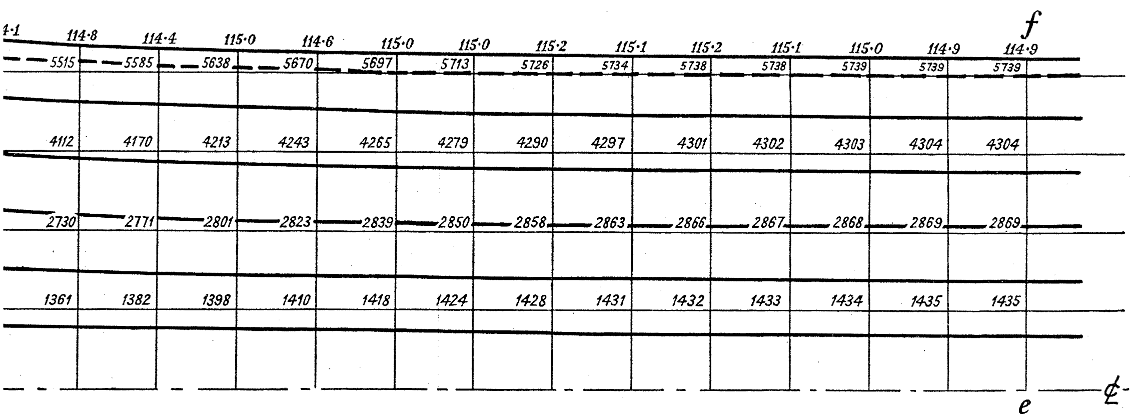

:ontours of 100ψ.

(*Facing p.* 132)

and hence, when $a^2/b^2 = 36$ as before,

$$c^2/b^2 = 0{\cdot}5035 \quad (c/b = 0{\cdot}7096). \tag{29}$$

As in § 20 the values 12 and 2 were given to a and b, respectively, in figure 7. Far upstream of B the velocity was assumed to be axial and uniform, so that as defined in § 5

$$\psi = r^2 \tag{30}$$

on the left of the computational field; and then in the parallel free stream (far downstream of B)

$$\psi = r^2 \times \frac{a^2}{c^2} = 71{\cdot}4965r^2, \quad \text{by (29)}. \tag{31}$$

Thereby ψ was made zero on the axis of symmetry and was given the value 36 at all points on the rigid boundary $AEFB$ and on the wanted free stream-line BD (figure 7); i.e. the value 36 was given to the constant in the first of (11). In the second of (11), k had the value 143 (very approximately), as can be seen by considering the parallel free stream, in which

$$k = \frac{1}{r}\frac{\partial\psi}{\partial\nu} \equiv \frac{1}{r}\frac{\partial\psi}{\partial r} = 142{\cdot}993, \quad \text{according to (31)}.$$

Thus the double boundary condition to be satisfied on the free boundary was

$$\psi = 36, \quad \frac{1}{r}\frac{\partial\psi}{\partial\nu} = 143, \tag{11) A}$$

ψ, r, ν having non-dimensional significance.

24. The basis of our computations was (22), § 14. This is satisfied by the expression (30), which gives

$$\psi_0 = \psi_1 = \psi_3 = r_0^2, \quad \psi_2 = (r_0+a)^2, \quad \psi_4 = (r_0-a)^2,$$

also by the expression (31). It has been shown to lead to a 'relaxation pattern' (figure 6) which is special to each value of the r-coordinate: on that account the labour of computation is much increased in this example as compared with Example 1, since the sharp curvature of the free stream-line close to B (figure 7) again makes it necessary to use a very fine net near that point.

It was remarked in § 14 that (22) requires modification when the grading entails square meshes inclined at 45° to the axes of z and r. Referring to figure 13, it is easy to deduce in such cases, as the relation corresponding with (22),

$$\mathbf{F}_0 = \psi_5+\psi_6+\psi_7+\psi_8-4\psi_0-\frac{a}{\sqrt{2}r_0}(\psi_2-\psi_4) = 0, \tag{32}$$

a having the significance which is indicated; and in conjunction with this we may, where convenient, use the relations $\quad 2\psi_2 = \psi_5+\psi_6, \quad 2\psi_4 = \psi_7+\psi_8.$

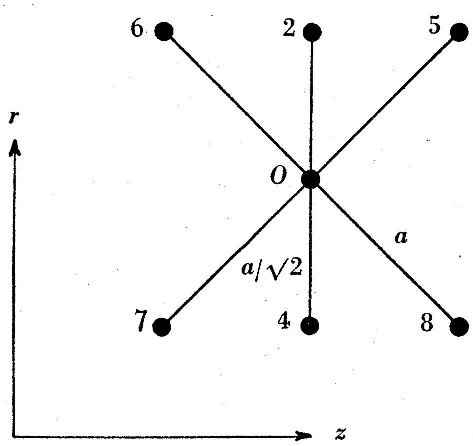

FIGURE 13

(More accurate interpolation formulae might be employed for ψ_2 and ψ_4 in the final stages; but usually this step is not necessary, for the reason that the cofactor of $(\psi_2-\psi_4)$, in (32), is small.)

25. Figures 14–16 exhibit values and contours of ψ as defined in § 5, multiplied (for avoidance of decimals) by 10, 100 or 1000 as stated in the appended legends. In figure 14, which shows the whole field of computation, the multiplier is 10 and the mesh-side a ranges, in the graded net, from $\frac{1}{2}$ to $\frac{1}{8}$; accepted values of ψ are exhibited in all parts of the field except a region close to B, part of which ($defg$) is shown to a larger scale in figure 15. There, the multiplier is 100 and a ranges from $\frac{1}{4}$ to $\frac{1}{32}$; accepted boundary values of $\frac{1}{r}\frac{\partial\psi}{\partial v}$ are recorded for comparison with the correct value $k = 143$ (§ 23). Again, in a region close to B, ψ-values are not given because the net was 'graded' still more finely, as shown in figure 16. There, the multiplier is 1000 and a ranges from $\frac{1}{32}$ to $\frac{1}{128}$; values of $1000\,\psi$ and contours of 10ψ are exhibited, also values of $\frac{1}{r}\frac{\partial\psi}{\partial v}$ on the free stream-line.

Example 3. *Orifice plate in a circular tube*

26. Here for the first time we confront a problem which it is not possible to formulate completely at the outset, for the reason that the coefficient of contraction cannot be predicted from momentum considerations in the manner of § 19. Computation must proceed by a process of trial and error, as described in § 6: adopting the second of the two alternatives there propounded, we assigned values to ψ on the centre-line and along the rigid boundary and free stream-line, then, starting from an assumed shape of free stream-line and an assumed value of k, computed the 'boundary error' η defined by

$$\eta = \frac{1}{r}\frac{\partial\psi}{\partial v} - k, \tag{33}$$

and reduced this (sensibly) to zero all along the stream-line by systematic modification *both of k and of the shape*. For every value of k the 'asymptotic solution'

$$\psi = \tfrac{1}{2}kr^2 \tag{34}$$

was taken as holding far downstream of B: thereby the second of (11) was satisfied far downstream, independently of the value given to the constant in the first. ψ_B denoting this constant (so that $2\pi\psi_B$ measures the total flow through the orifice) and ρ^2 the 'coefficient of contraction' (so that ρ is the 'asymptotic radius' when the radius of the orifice = 1), we had, according to (34),

$$k\rho^2 = 2\psi_B, \tag{35}$$

and hence (knowing ρ approximately from hydraulic experiments) we could after fixing ψ_B make a good initial estimate of k.

27. Clearly, in order that η may be given with precision, the gradient of ψ needs to be estimated closely, so again the final net must have small meshes in the neighbourhood of B. Figure 17 exhibits the complete field of computation, and shows the mesh-side a graded from $\frac{1}{2}$ (in parts remote from B) to $\frac{1}{16}$ inside the region $BCDEFGB$. In figure 18 this region has been redrawn to an enlarged scale, and again the net is graded from $a = \frac{1}{4}$ to $a = \frac{1}{64}$ in a small region $BabcdeB$ surrounding B. Finally, the solution within this last region is exhibited in figure 19. Along the free stream-line in figures 18 and 19, numerals just outside the stream record computed values of $\frac{1}{r}\frac{\partial\psi}{\partial v}$ and of η as defined in (33). The accepted value

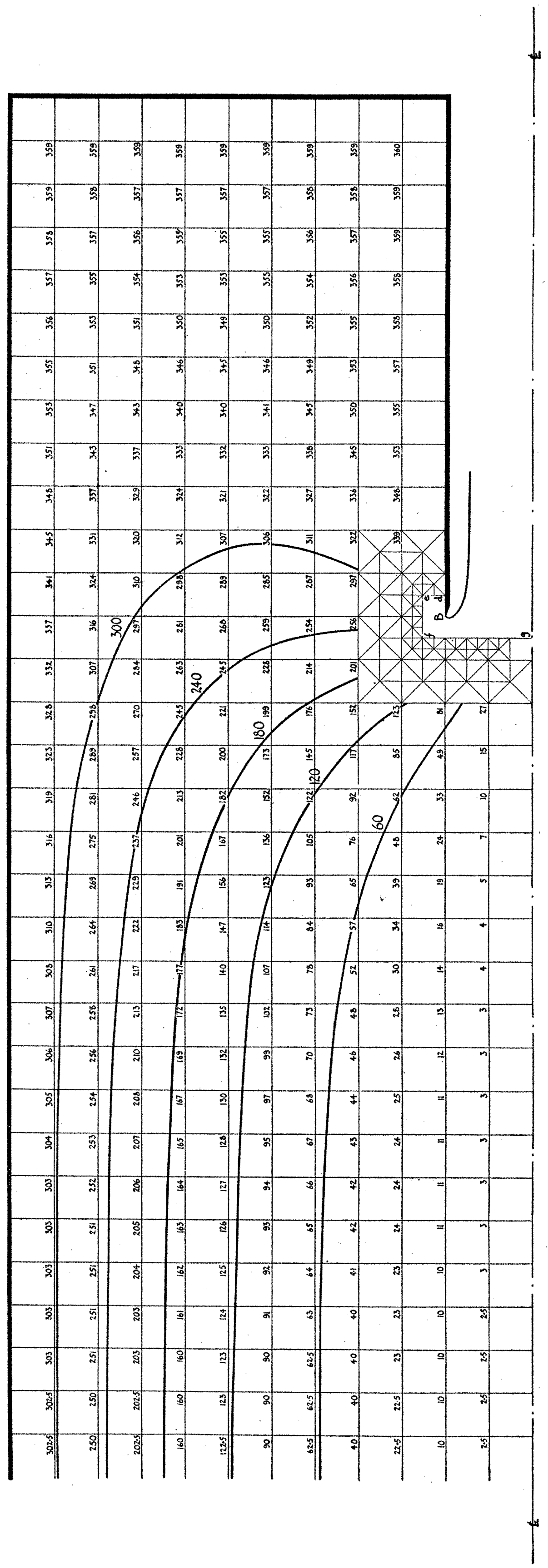

FIGURE 14. Values and contours of 10ψ.

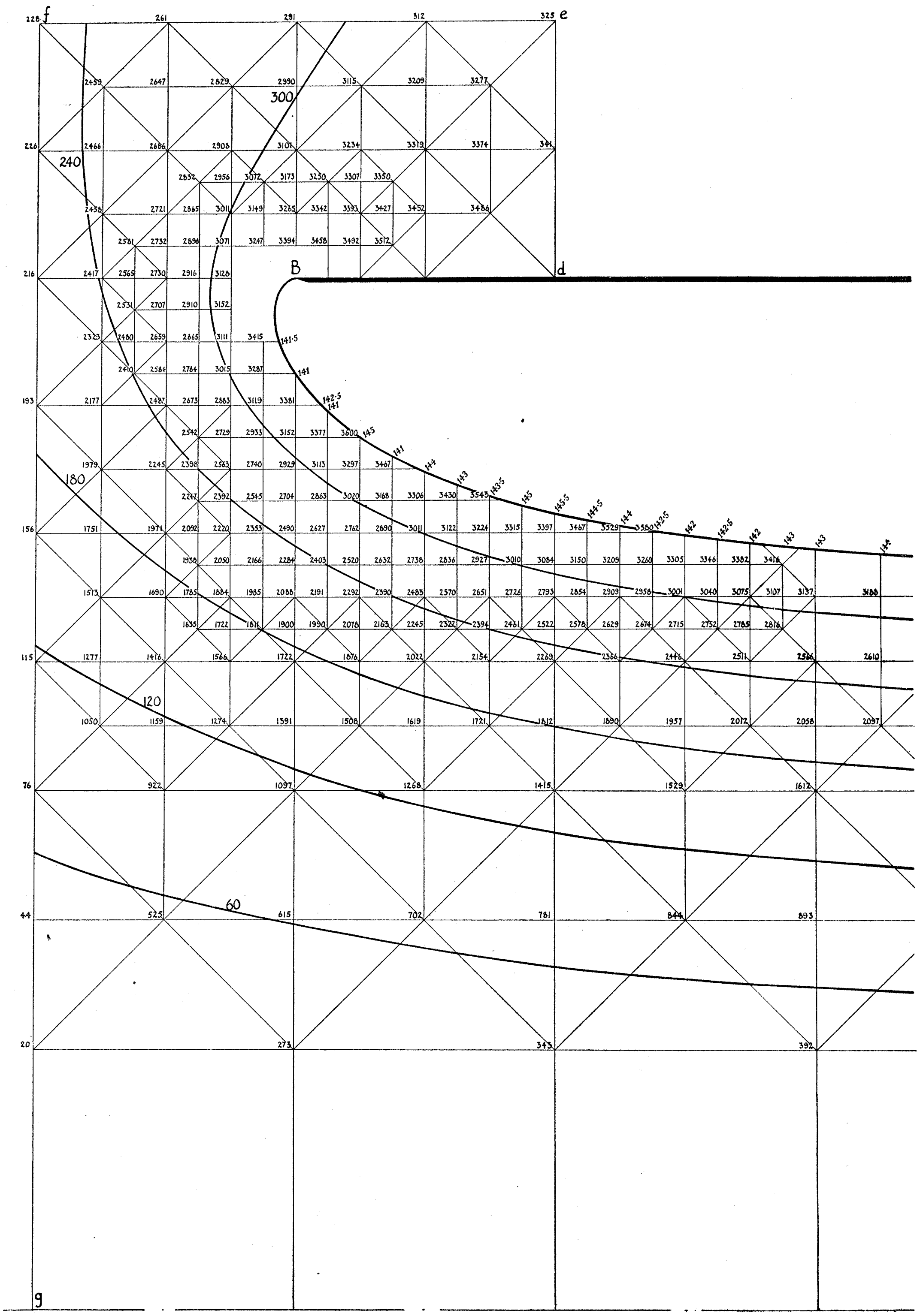

FIGURE 15A. Values of 100ψ and contours of 10ψ.

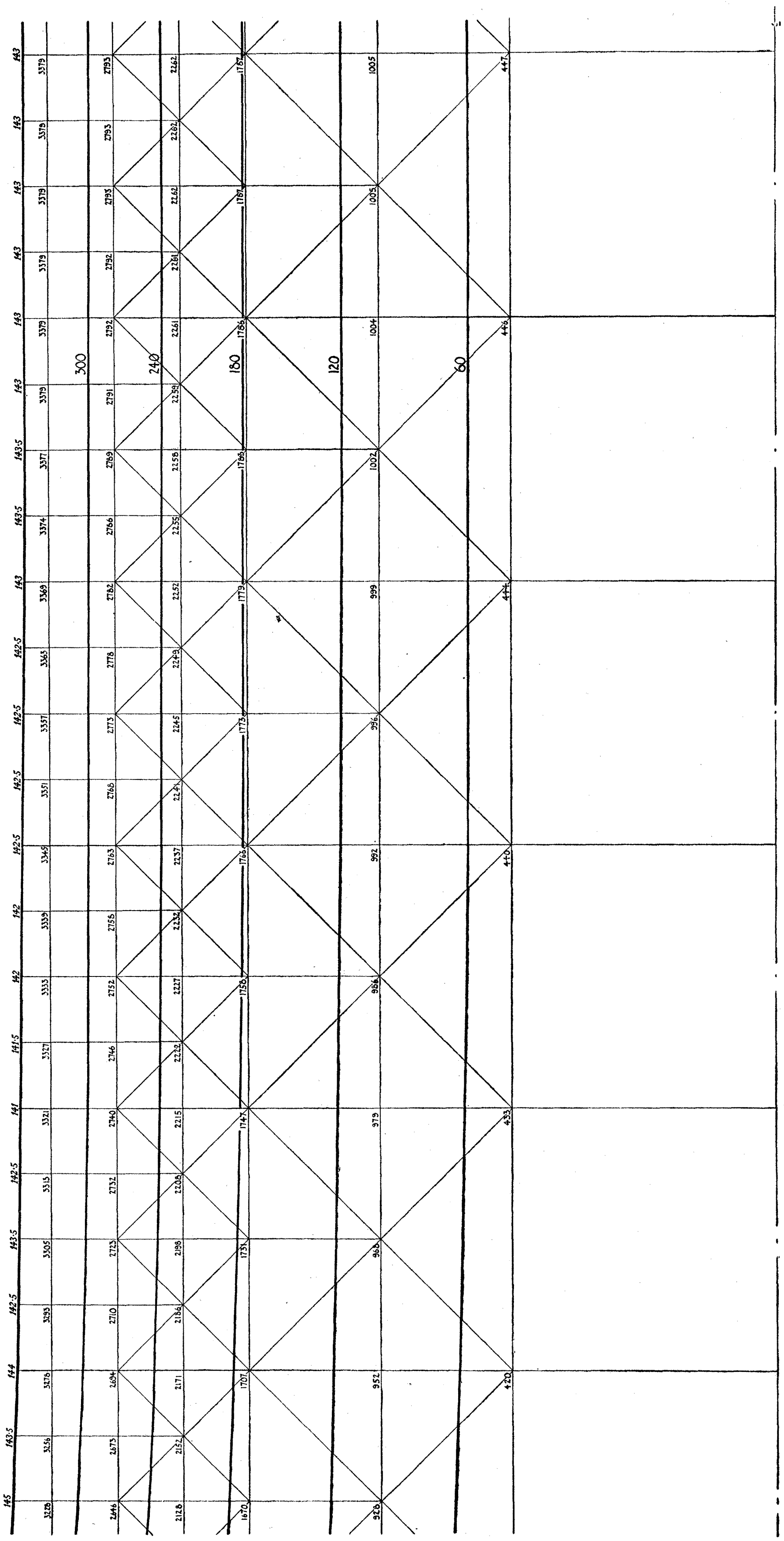

Figure 15B. Values of 100ψ and contours of 10ψ.

of k was 118·34, corresponding with a coefficient of contraction $\rho^2 = (0{\cdot}78)^2 = 0{\cdot}6084$.* Having this value we could use the method of § 19 to compute the total pressure on the orifice plate.

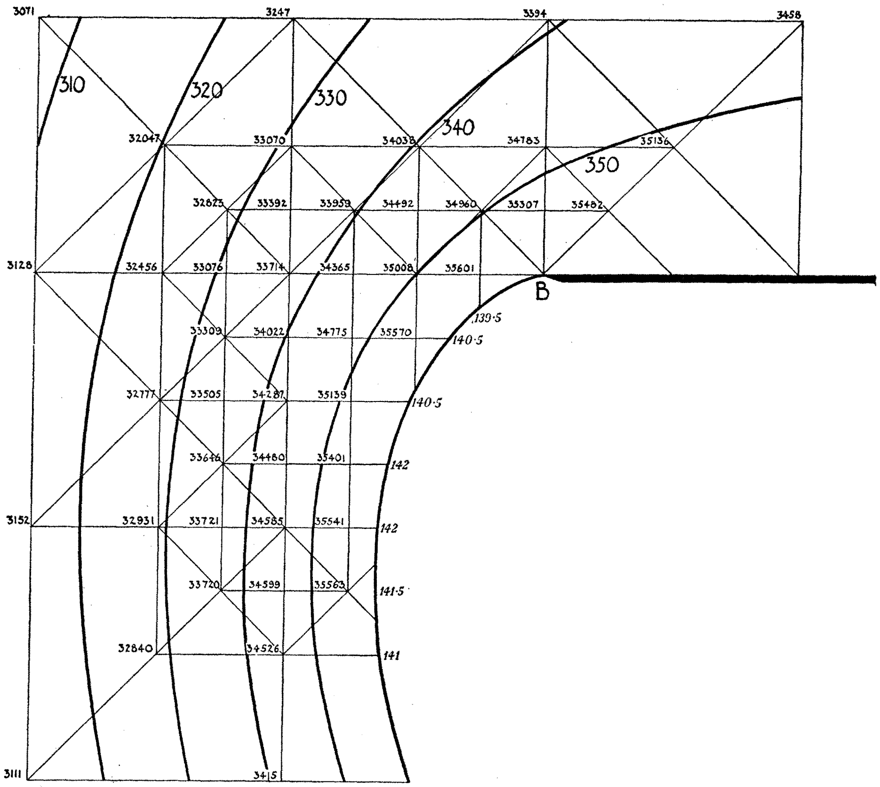

FIGURE 16. Values of 1000ψ and contours of 10ψ.

28. Here again computation was made more laborious by the occurrence of r in the 'relaxation pattern'. The nature of the requisite modification was never in doubt; for an increase in the assumed value of k, unaccompanied by any change in the assumed shape, decreases η everywhere according to (33), while an increase given at some point to the radius of the jet entails a reduction at that point of $\dfrac{\partial\psi}{\partial\nu}$ and therefore of $\dfrac{1}{r}\dfrac{\partial\psi}{\partial\nu}$ and η. At neighbouring points it increases η by rendering the jet more concave: consequently errors of opposite sign could be eliminated by changing the shape of the free stream-line, errors all of like sign by changing k.

In fact only two changes of k were necessary, since the second (by good fortune) led to negligible errors at all points. Initially ρ was assumed to have the value 0·71, and subsequently this was altered to 0·75 and 0·78 (the accepted figure). Our first attempts were made on a net fairly coarse in all parts, since it was apparent that a finer mesh near B would yield for that point a value of $\partial\psi/\partial\nu$ higher by an amount fairly easy to estimate.

One point perhaps calls for notice:—In § 26, by taking $2\pi\psi_B$ as a measure of the total flow, we gave 'dimensional' significance to ψ, whereas in the computations r, z, ν, a were (as is necessary) treated as 'non-dimensional'. But the point has no importance in this example, because the boundary conditions, namely, (11) of § 5, are still satisfied when ψ is multiplied by any constant quantity (the constants on their right-hand sides being multiplied similarly).

* '...this coefficient is found experimentally to be about 0·62' (Lamb 1932, § 24).

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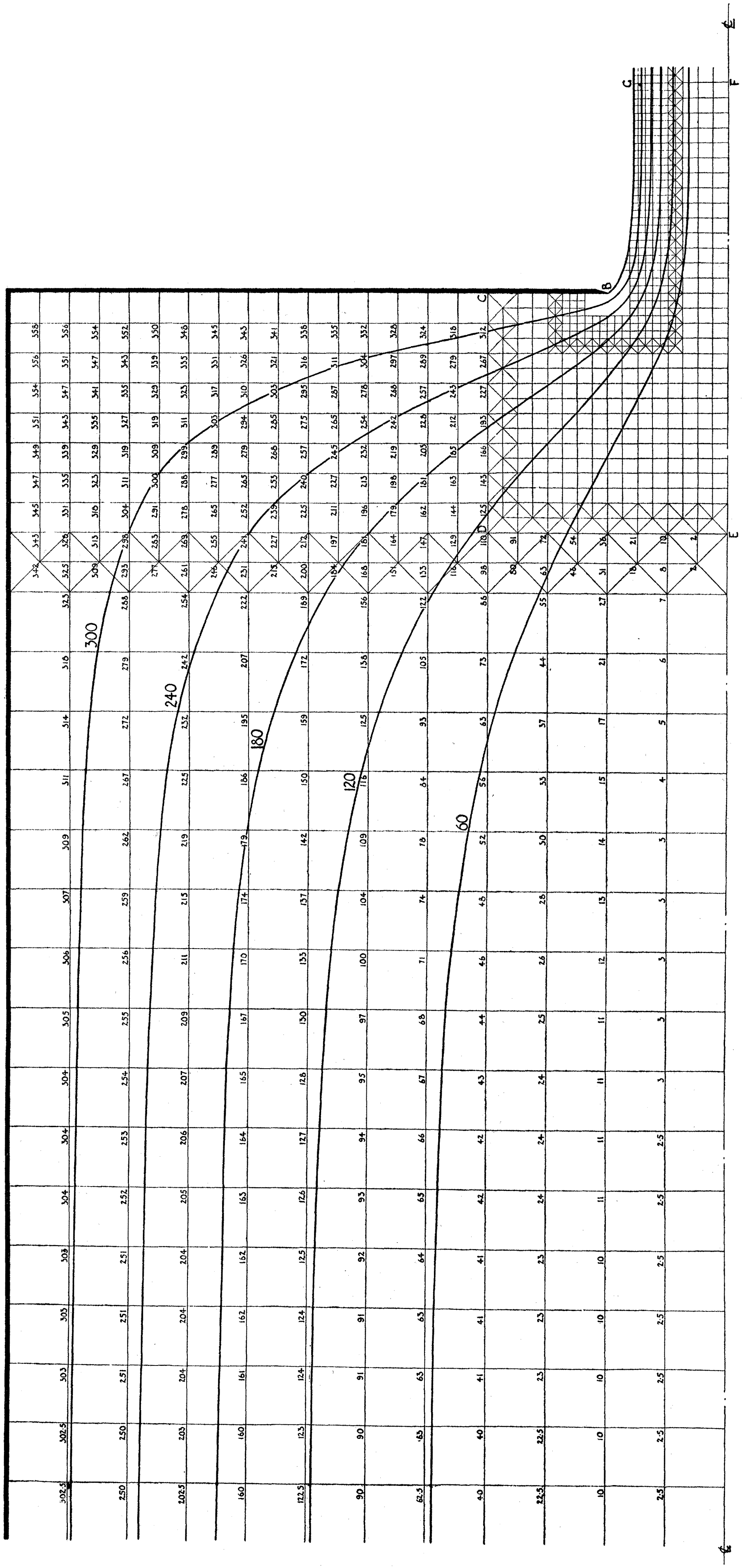

FIGURE 17

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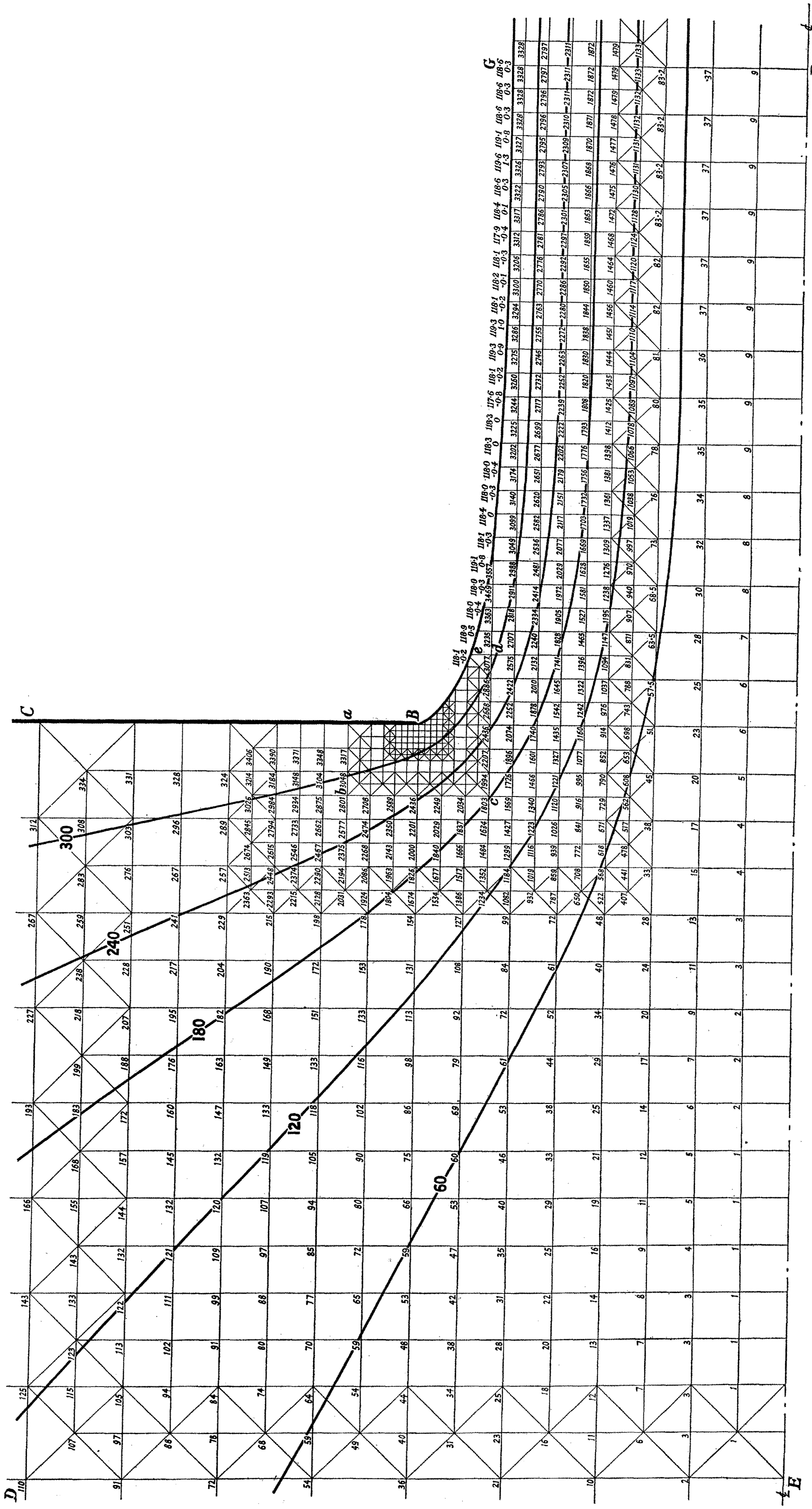

FIGURE 18

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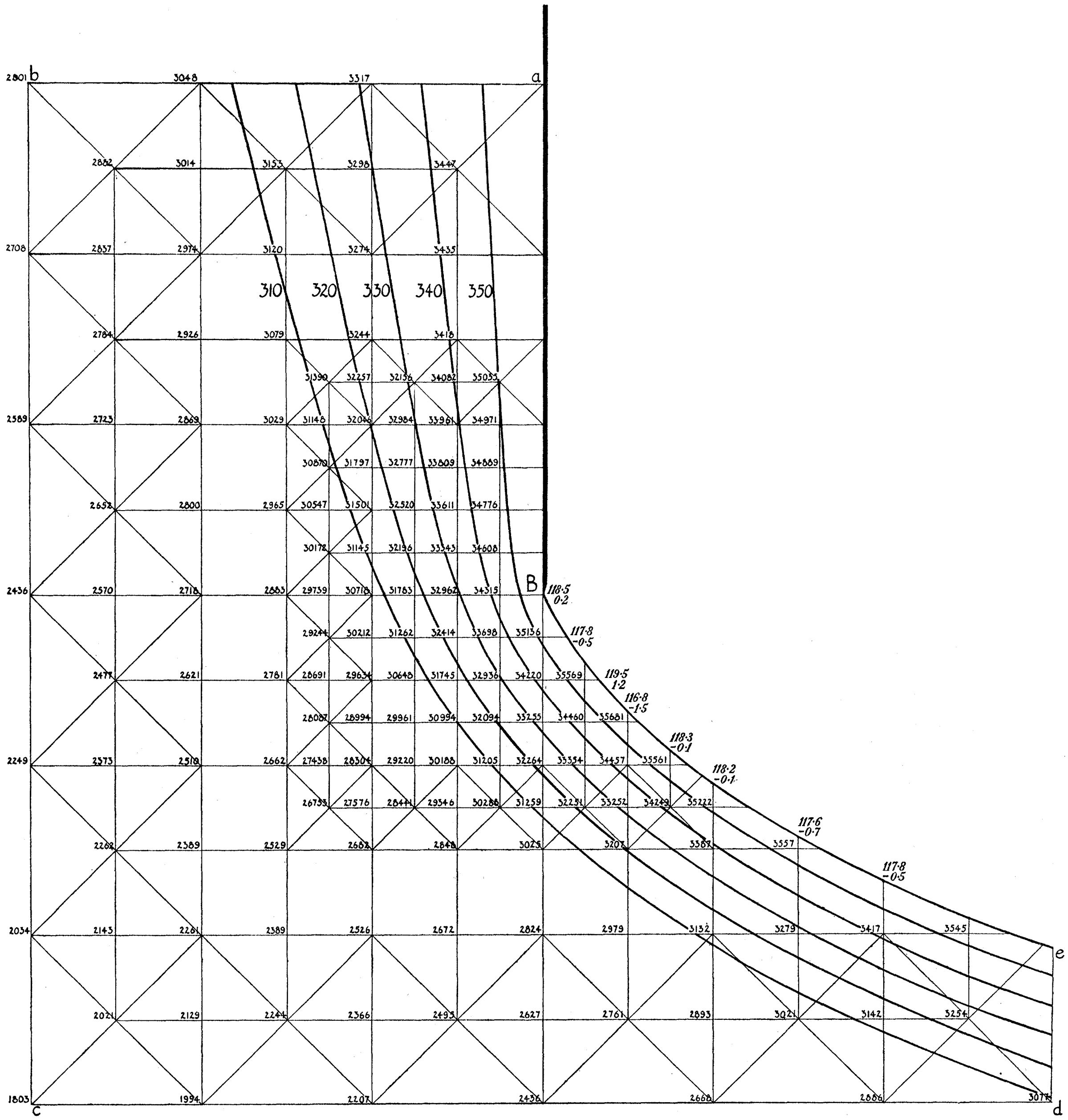


FIGURE 19

Example 4. *Flow under gravity through an orifice plate*

29. In our next example the circumstances are different for the reason that the flow is maintained by gravity operating in the axial direction. The governing equation (10) is associated with boundary conditions as stated in (16), § 7, where ψ carries the non-dimensional significance attached to ψ' in (14). The relative dimensions of the tube and orifice are the same as in Example 3, and physical considerations suggest that the stream-lines will not be altered sensibly, except close to the orifice and in the 'free' jet.

Figure 20 confirms this expectation. The new stream-lines are shown in full line, while broken curves (reproduced from figures 17–19) show the stream-lines computed for Example 3. Figure 21 exhibits finer detail in the neighbourhood of B.

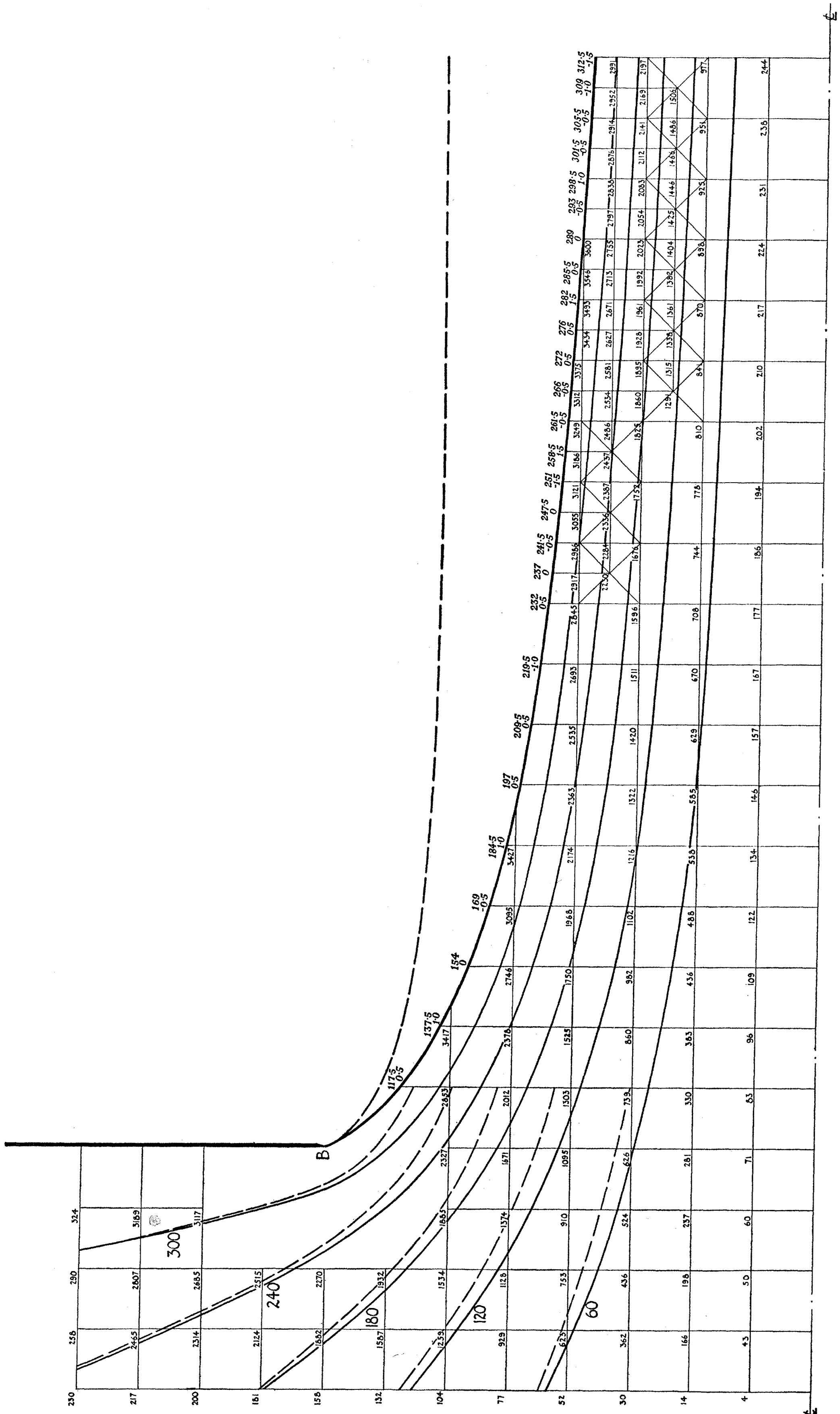


FIGURE 20

30. Few features call for special notice. ψ_B denoting (as in § 26) the value taken by ψ along the rigid boundary and free stream-line, the double boundary condition (16) may be written in the form

$$\psi = \psi_B, \qquad \frac{1}{r}\frac{\partial \psi}{\partial \nu} = \sqrt{(z+k)}, \tag{36}$$

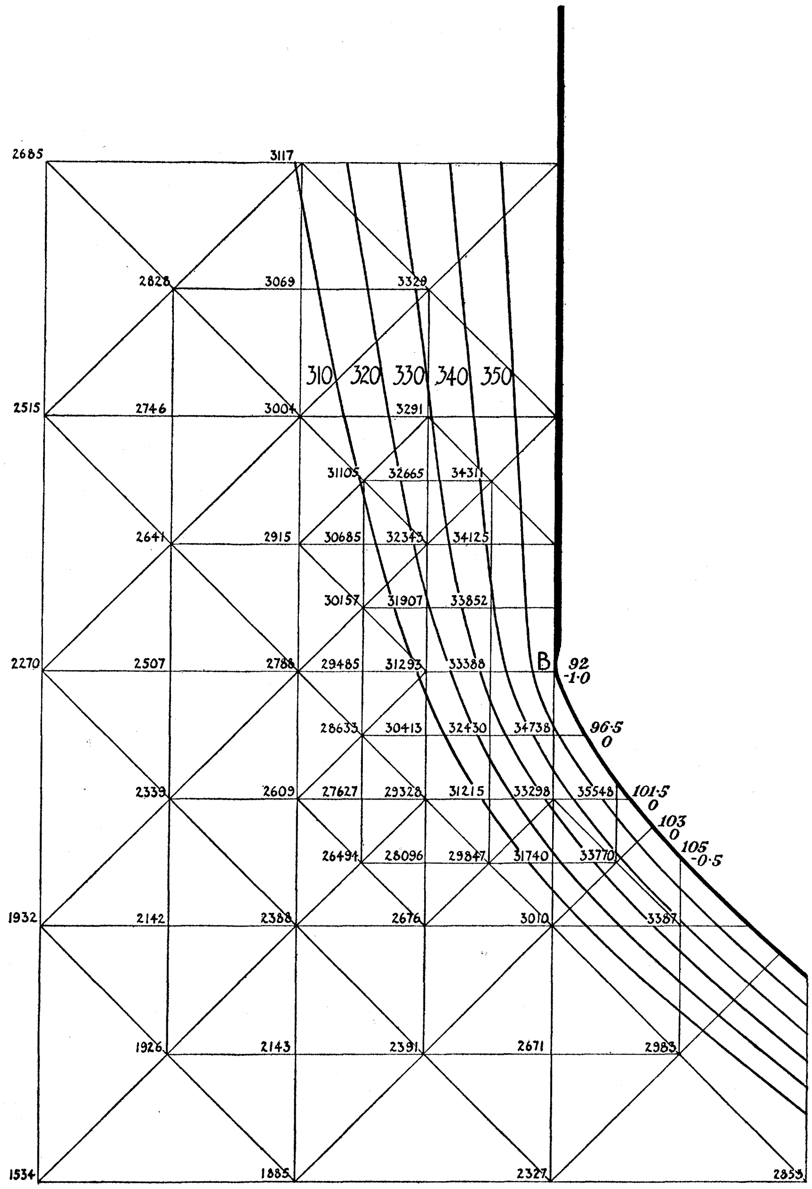


FIGURE 21

all symbols having non-dimensional significance, and k being a constant which must be determined in the course of computation. As before we substitute their finite-difference approximations for (36) and for the governing equation (10).

Far downstream of the orifice (assuming the jet to have small taper) we may for a first approximation take ψ to vary as r^2 and identify $\partial\psi/\partial\nu$, in the second of (36), with $\partial\psi/\partial r$. On that understanding the second of (36) requires that

$$\psi = \tfrac{1}{2}r^2\sqrt{(z+k)}, \quad \psi_B = \tfrac{1}{2}R^2\sqrt{(Z+k)}, \tag{37}$$

R and Z denoting co-ordinates of a point on the jet's surface. Any arbitrary value may be given to ψ_B, and with this can be associated a trial value for k: then from the second of (37) we can deduce a corresponding first approximation to the profile of the jet downstream, continue this profile to B with a use of Relaxation Methods, and repeat the process (with different values assumed for k) until the double boundary condition has been satisfied at all points of the free surface.

Section II. Jets conditioned by gravitational forces

Example 5. Free jet falling under gravity (waterfall)

31. So far, in all of our examples the jet has had only one 'free surface'—or at least could be treated on that basis, because a line of symmetry could be taken as fixed. In our next example the jet has two free surfaces, fluid being assumed to flow along a horizontal bed to a point at which (figure 22) it springs clear and continues (as a 'waterfall') under the influence of gravity alone. If gravity were inoperative it would continue as a horizontal stream (dotted in figure 22), which accordingly is the initial solution requiring to be modified. For such modification, in the preliminary stages, Method (B) of § 13 was employed.

Following that method, we apply edge-loadings given by the second of (15) to the upper and lower surfaces of the dotted jet; and the resulting stream-lines ($\psi = 0$ and $\psi = 2000$) bend downwards from A as they should. We then apply appropriate loadings to these altered stream-lines, with the result that they deflect further; and so on, until the solution has converged. Of course, here as in any application of Method (B) one must be on guard against the latent instability which was noticed in § 17, and for that reason Method (A) was introduced after one or two stages of computation.

32. Again referring to the dotted jet in figure 22, we observe that in the application of Method (B) not only its horizontal surfaces, but also the vertical section (chosen arbitrarily) which limits the field of computation, must be given 'edge-loadings' in accordance with the second of (15). The total loading on the vertical section must be such as will satisfy a condition imposed at A: its distribution (as in 'Saint Venant's principle') is immaterial provided that the section lies well to the left of that point.

The condition imposed at A is that there (since the jet becomes 'free') the velocity—i.e. the normal gradient $\partial\psi/\partial\nu$—must have a value given by the second of (15). Within the accuracy of the finite-difference approximations it can be satisfied by a synthesis of two trial solutions, because the principle of superposition is applicable. Suppose that on applying two different loads W_1, W_2, in turn, to the vertical section we arrive at gradients $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, respectively, at A, and that the gradient imposed by (15) has a value $\boldsymbol{g}$: then the wanted gradient can be obtained by combining the two solutions multiplied by α and β respectively, where

$$\alpha \boldsymbol{g}_1 + \beta \boldsymbol{g}_2 = \boldsymbol{g}; \tag{i}$$

and the edge-loadings on the horizontal surfaces will have the intensities required by (15) provided that

$$\alpha + \beta = 1, \tag{ii}$$

because they have these intensities in each solution severally.

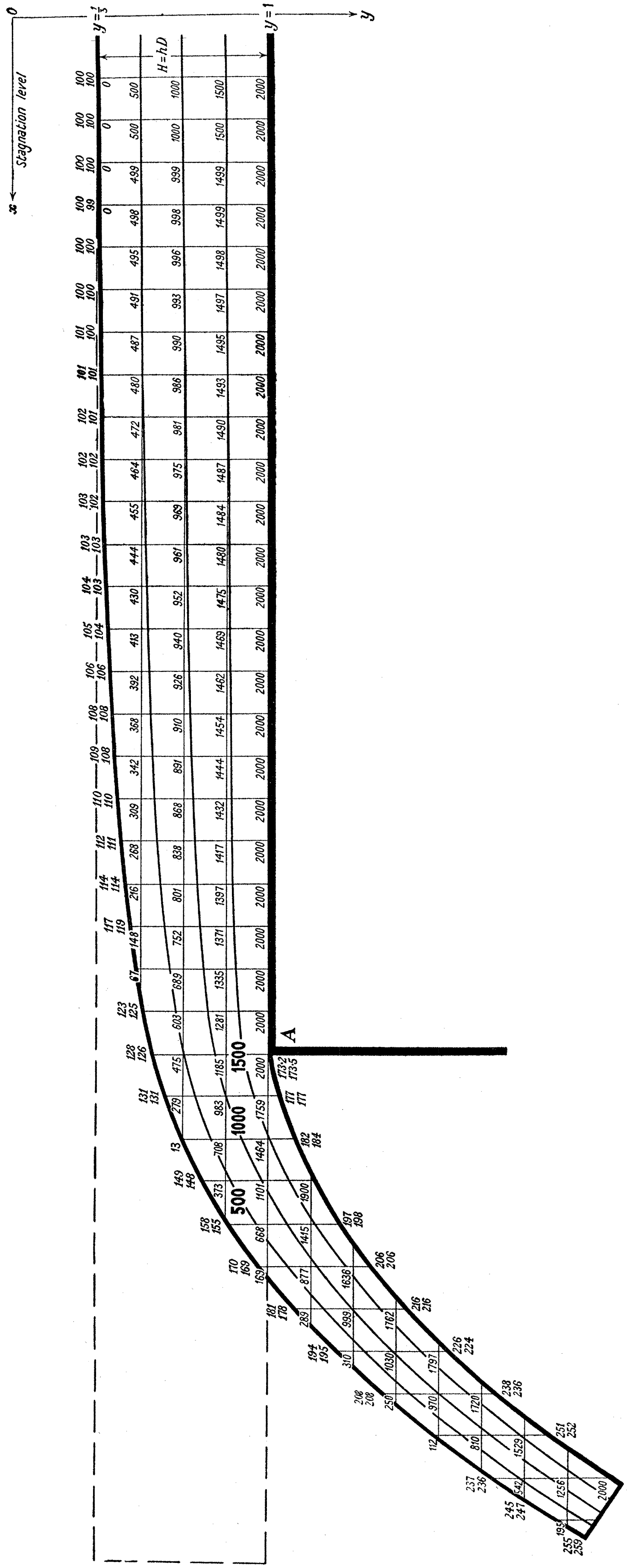


FIGURE 22

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Combining (i) and (ii), we deduce that

$$\alpha = \frac{g - g_2}{g_1 - g_2}, \quad \beta = \frac{g_1 - g}{g_1 - g_2}, \tag{iii}$$

whence the required solution can be obtained.

33. Suppose the stagnation level to be specified, and let D in (13) stand for the distance between this level and the horizontal bed. Far to the right of A (figure 22) let $H = hD$ be the depth of the (uniform) stream, so that y (non-dimensional) has the value $(1-h)$ at the free surface, 1 at the rigid bed; and let the corresponding values of ψ be 0 and C. Then the second of (15), viz.

$$\frac{\partial \psi}{\partial \nu} = \sqrt{y}, \tag{38}$$

gives far upstream of A (where the stream is uniform and horizontal)

$$\frac{C}{h} = \sqrt{(1-h)}. \tag{39}$$

Hence C is known when h is specified, and attains its maximum value $(2/3\sqrt{3})$ when $h = \frac{2}{3}$. This case was chosen for study (cf. figure 22); but in computation (to avoid decimals) we gave ψ the value 2000 at the rigid bed and on the lower free stream-line, and made allowance for the multiplying factor $(3000\sqrt{3})$ when comparing normal gradients with the values required by (38).

Figure 22 records our accepted solution. Since (38) requires the boundary gradient to have different values at different levels, italic numerals give, at points where 'strings' intersect the free stream-lines, computed values of $\sqrt{y}$ (above) and of $\partial\psi/\partial\nu$ (below), both multiplied by $100\sqrt{3}$. The discrepancies are nowhere large.

Example 6. *Stationary waves of finite amplitude in a stream of finite depth*

34. A very important branch of hydrodynamics is concerned with the oscillations under gravity of a liquid having a free surface.* In certain cases these combine to form 'progressive waves' which travel over the surface, in one direction only, without change of form. Then, if we impress on the whole mass of liquid a velocity equal and opposite to that of propagation, they present a problem—of stationary waves in a uniform stream—to which the methods of this paper are applicable.

The orthodox theory of waves is difficult, except when simplified by restrictions (e.g. to waves of *small* amplitude) which render the equations linear. Rayleigh (1876*a*, 1911) showed that the exact solution *for the case of infinite depth* is contained in the formulae

$$\phi/c = -x + \beta e^{ky} \sin kx, \qquad \psi/c = -y + \beta e^{ky} \cos kx, \tag{40}$$

and confirmed an approximation, due to Stokes (1847), in which the equation of the wave-profile is

$$y = \text{const.} + a\{\cos kx - (\tfrac{1}{2}ka + \tfrac{17}{24}k^3a^3) \cos 2kx + \tfrac{3}{8}k^2a^2 \cos 3kx - \tfrac{1}{3}k^3a^3 \cos 4kx + \ldots\} \tag{41}$$

* This section is based on Lamb (1932), chapters VIII and IX.

and the wave-velocity c (i.e. the stream-velocity for which the wave is stationary) is related with its amplitude (defined by a) and with its wave-length ($\lambda = 2\pi/k$) by the expression

$$c^2 = \frac{g}{k}(1+k^2a^2+\tfrac{5}{4}k^4a^4+\ldots). \tag{42}$$

According to (42) the stream velocity c, for given λ, increases steadily (but at first slowly) as the ratio of amplitude to wave-length (measured by ka) increases from zero. *When the amplitude is infinitesimal*, c is related with λ by

$$c^2 = g/k = g\lambda/2\pi. \tag{43}$$

In regard to the effect of stream-depth (h), a treatment restricted to infinitesimal amplitudes (Lamb 1932, § 229) shows that (43) may be generalized in the form

$$c^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}, \tag{44}$$

which differs negligibly from (43) when $h > \frac{1}{2}\lambda$, and only slightly (c being reduced by less than 3 %) when $h > \frac{1}{3}\lambda$.

Since $\tanh x \approx x$ when x is small, when h/λ is small (44) is closely represented by

$$c^2 = gh, \tag{45}$$

i.e. c depends on h alone, and is the same for all wave-lengths; so there is a unique stream-velocity with which stationary waves can be associated when the length of these is great in relation to the stream-depth.

We took as our next problem the determination of wave profiles for stationary waves of finite amplitude in a 'canal' of limited depth.

35. Some preliminary indications can be drawn from the results just cited. In our notation—D denoting (as in § 33) the depth below stagnation level of the horizontal bed, and $H = hD$ the depth of the (uniform) undisturbed stream—

$$c^2 \text{ in § 34} = 2g(D-h) = 2gD(1-h) \text{ in our notation,}$$

$$h \text{ in § 34} = H = hD \text{ in our notation.}$$

If then λ, in § 34, is replaced by $l.D$ (so that l denotes the 'non-dimensional wave-length'), equations (43), (44) and (45) are replaced by

$$l = 4\pi(1-h), \tag{46}$$

$$l = 4\pi(1-h)\coth\frac{2\pi h}{l}, \tag{47}$$

and

$$3h = 2, \tag{48}$$

respectively. *All three of these equations relate to waves of infinitesimal amplitude.* For deep-water waves, equation (46) relates l with $(1-h)$—the non-dimensional depth below stagnation level of the undisturbed free surface; (47) introduces the effect of finite depth of stream; (48) shows that l can have any value large in relation to h when $h = \frac{2}{3}$.*

36. In our investigation, h (non-dimensional) was given a fixed value 11/12,—i.e. a constant depth and velocity were postulated for the undisturbed stream; and wave-profiles

* Then (cf. § 33) the flow has the maximum value which is compatible with specified positions of the stagnation level and of the horizontal bed.

were determined for a series of chosen amplitudes, the wave-length being allowed in each instance to emerge as a result of computation,—i.e. varied until an acceptable solution was obtained. As in § 33 the total flow C can be related with h by (39): when $h = 11/12$,

$$C = 11/24\sqrt{3},$$

so ψ (non-dimensional) was given the values 0 on the horizontal bed, $11/24\sqrt{3}$ at the free surface (the stream being assumed to move from left to right). For convenience, in the actual computations values 0 and 11 ($\times 10^4$) were taken—i.e. an arbitrary multiplying factor $24\sqrt{3}$ ($\times 10^4$) was employed.

A new feature is the existence of side boundaries which must be moved as the work proceeds—namely, the verticals through crest and trough, on which (since they are lines of symmetry) $\partial\psi/\partial x = 0$. Their distance apart is $\frac{1}{2}l$, the quantity which is to be determined for an assumed height of crest. The governing equation is (8) of § 4, and the boundary conditions are

$$\left.\begin{aligned} &\frac{\partial\psi}{\partial\nu} = 0 \text{ on the vertical side boundaries,} \\ &\psi = 0 \text{ on the horizontal bed,} \\ &\text{with (15), written in the forms} \\ &\left.\begin{aligned} &\psi = 11/24\sqrt{3}, \\ &\eta = -\sqrt{y} + \frac{\partial\psi}{\partial\nu} = 0, \end{aligned}\right\} \text{ on the free stream-line.} \end{aligned}\right\} \quad (49)$$

As in previous examples, the 'error' η decided the course of a computation: its sign determined the direction in which a boundary should be moved, its magnitude indicated the amount of the required displacement. Usually it was corrected, near a crest or trough by moving the vertical side-boundaries, elsewhere by altering the shape of the assumed free stream-line. At nodal points on or near the vertical side boundaries, special 'relaxation patterns' were employed to take account of the evanescence, at such boundaries, of $\partial\psi/\partial x$.

37. Figure 23 presents our accepted wave-profiles for the cases investigated. Figure 24 relates the non-dimensional wave-length l with the non-dimensional crest-height A (i.e. the height of the wave-crest above the level of the undisturbed stream) for values of A ranging between 0 and 1/12 (at which value a cusp at stagnation level replaces the smooth wave-crest). Within that range l falls from 1·11 to 0·88.

The higher value does *not* agree with the value predicted by equation (47), and we have not yet succeeded in accounting for the discrepancy. Though small (of the order of 6 %), it can hardly be due to errors of computation, in view of the consistency of the plotted points in figure 24. In fact quite unusually high standards of accuracy were imposed in these computations, as is shown by the pairs of numerals appended to the wave profiles in figure 23. (The upper numeral of each pair gives the computed value of $\partial\psi/\partial\nu$, the lower gives the 'error' η defined in § 36.) Our reasons were (1) the tendency of errors to cancel and so to remain undetected near side-boundaries, (2) our desire to determine accurately (and for the first time) the nature of the variation of l with A. Obviously this last objective would have been attained more easily if l had varied more rapidly.

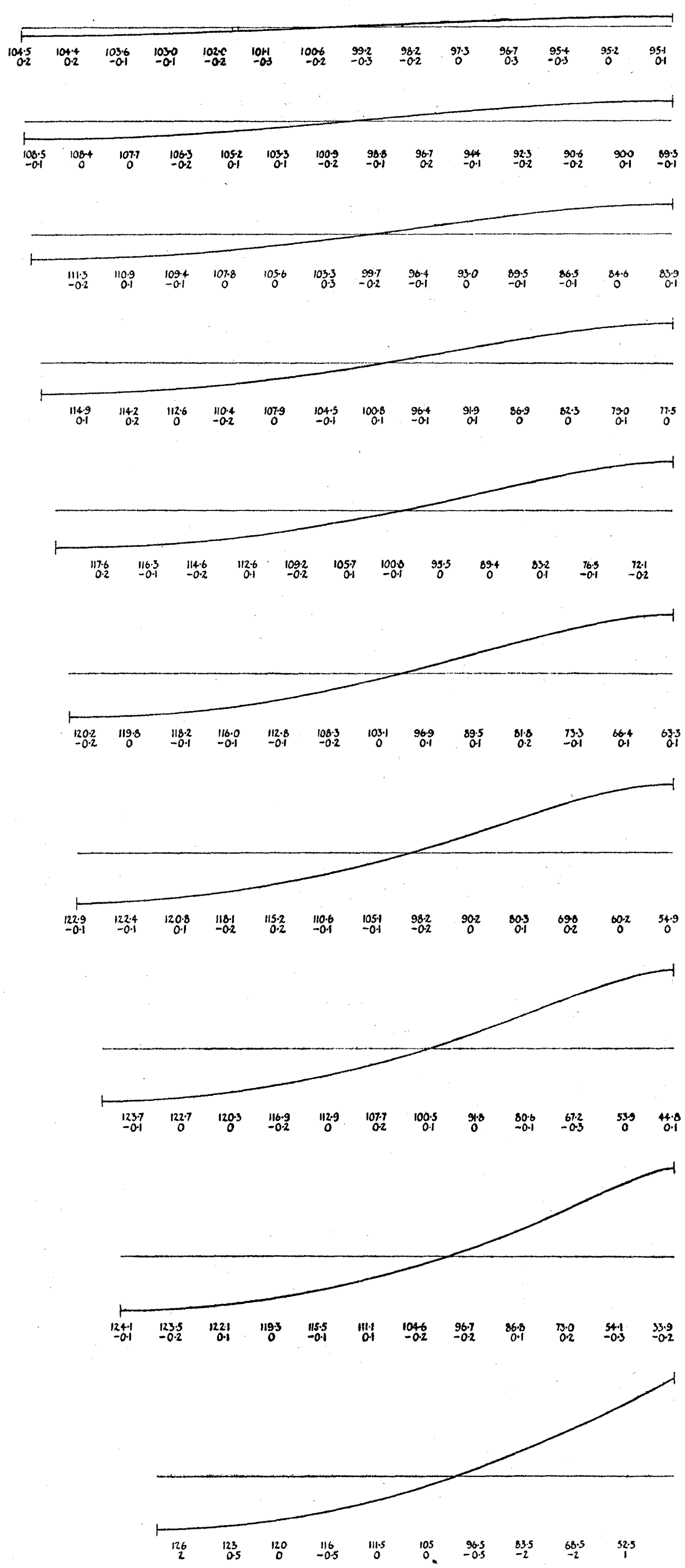


FIGURE 23

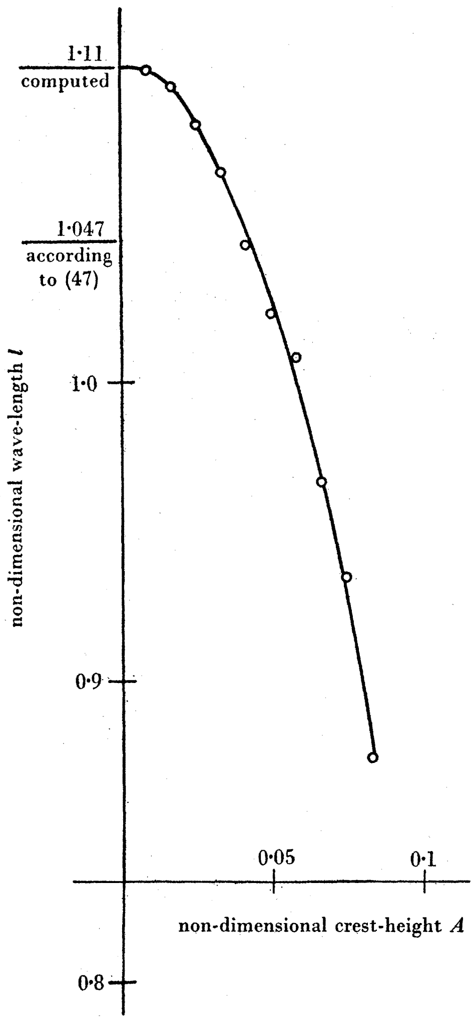


FIGURE 24

Example 7. *Flow of water under a 'drowned sluice'*

38. This example is similar to Example 3 (§ 26) in that either the total flow or the sluice-opening S (figure 1) must be left to be determined by computation. We chose the second alternative.

As shown in § 10, two depths of parallel stream can consist with given values of the total flow and of the depth below 'stagnation level' of the rigid horizontal bed: in theory either is conceivable far downstream of the sluice. Thus one must expect that a symmetrical solution exists in which the free surface regains its original level after passing the sluice. But flow of this kind would entail high or infinite velocity at A in figure 1 (according as the sluice has small or zero thickness), therefore (in practice) negative pressure or 'cavitation' close to A. The one solution having practical interest is that which is indicated in figure 1, unless a regime is possible in which waves are generated in the issuing stream. We shall return to this question in § 42.

39. In the membrane analogue, according to § 10, unstable conditions exist upstream, where the circumstances correspond with A in figure 2, but stable conditions downstream, where the circumstances correspond with B. Neither region in fact gave trouble, except that very fine grading had to be employed, as indicated in figure 25, to determine S with precision. The assumed conditions upstream were those of § 36, and a multiplying factor $24\sqrt{3} \times 100$ was employed. On the finest net (part of which is reproduced in a supplementary diagram) ψ-values were computed to one more significant figure.

The accuracy of the accepted solution is shown by the close agreement of the recorded boundary values of $\sqrt{y}$ (above) and of $\partial\psi/\partial\nu$ (below).* Downstream the free surface falls continuously, the 'asymptotic depth' (which is calculable for the assumed conditions) being 0·66 of the computed opening S.

Example 8. *Two-dimensional planing surface* (*seaplane*)

40. Giving the rigid boundary (sluice) in Example 7 a 30° instead of 90° inclination to the horizontal, we have the problem indicated in figure 26,—the two-dimensional case of a planing surface such as is used in hydroplanes and flying boats. By reason of this practical application the problem has received much attention. A full review of work done up to 1934 was given in that year by K. W. Wagner: more recently the problem has been studied by A. E. Green (1935).

Because inclusion of gravity makes the problem intractable by the theory of a complex variable (§ 1), it was neglected both by Green and in most of the papers cited by Wagner, who asserts that it is unimportant when the planing angle is small. But solutions obtained on this basis have the characteristic that the oncoming stream bifurcates into two parts separated (at the planing surface) by a 'stagnation point' of zero velocity and full dynamic pressure: one part passes on, under the planing surface; but the other (termed the 'splash' or 'spray') is projected forward with a velocity which (since its free surface is at atmospheric pressure) has altered direction but unchanged magnitude and goes (in the theoretical solution) to infinity. Now because the 'splash' entails a continuous generation of momentum, of necessity it entails a downstream force on the planing surface, described as 'splash resistance'; but when gravity operates to prevent the free surface rising above 'stagnation level'

* Both quantities have been multiplied by $100\sqrt{12}$.

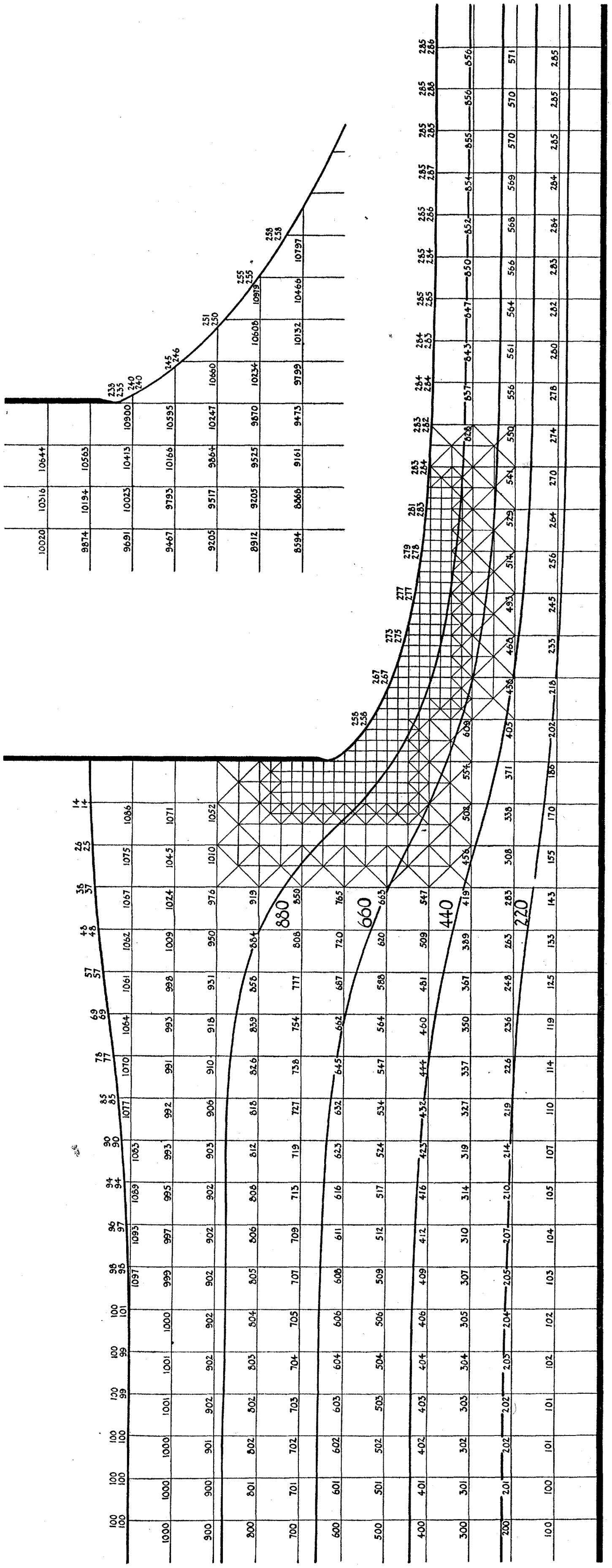


Figure 25

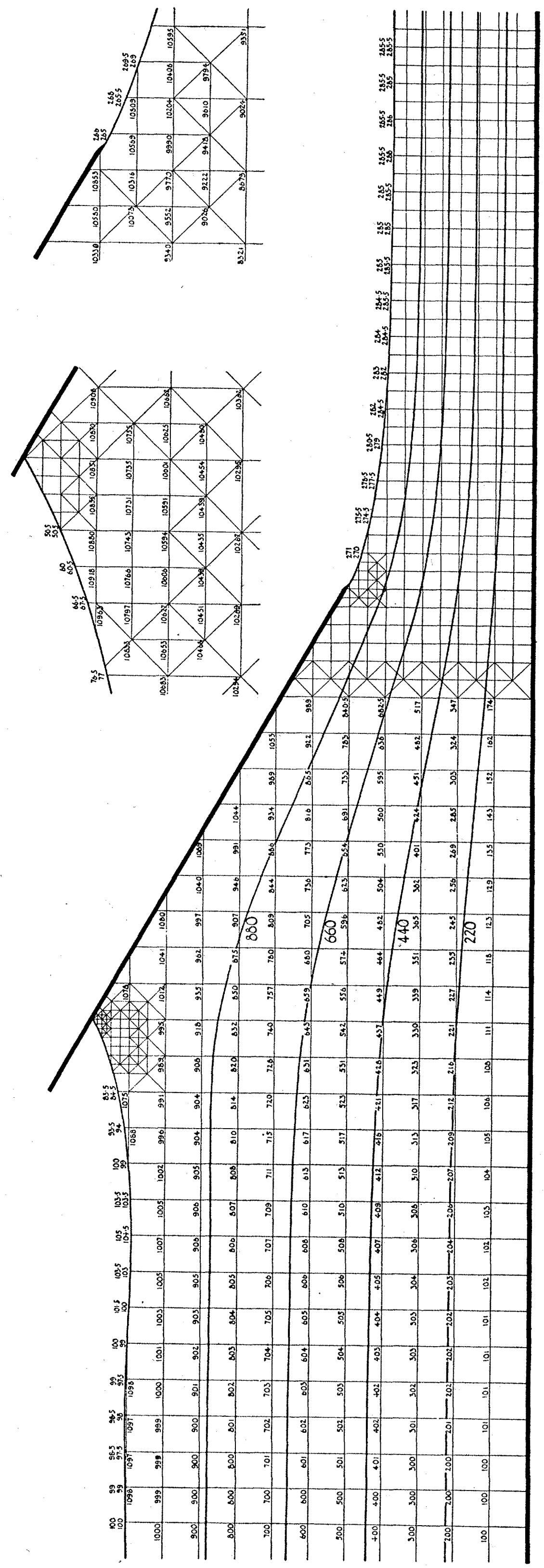


Figure 26

(§ 3), no such generation of momentum occurs. It would seem, then, that 'splash resistance' has no reality, and that a treatment which takes account of gravity must yield different *and more accurate* expressions for the forces on the planing surface. Accordingly we have applied such treatment to an example of the kind discussed by Green.

41. Figure 26 presents our accepted solution, the accuracy of which is indicated by the agreement of the recorded boundary values of $\sqrt{y}$ (above) and of $\partial\psi/\partial\nu$ (below).* The upstream conditions of Example 7 were assumed, and the gap between planing surface and horizontal bottom (S, § 38) was deduced by computation; a multiplying factor $24\sqrt{3} \times 100$ was employed, and here too the net had to be graded in order to determine S with precision. On the finest nets (parts of which are reproduced) ψ-values were computed to one or two more significant figures.

The question of wave-formation downstream

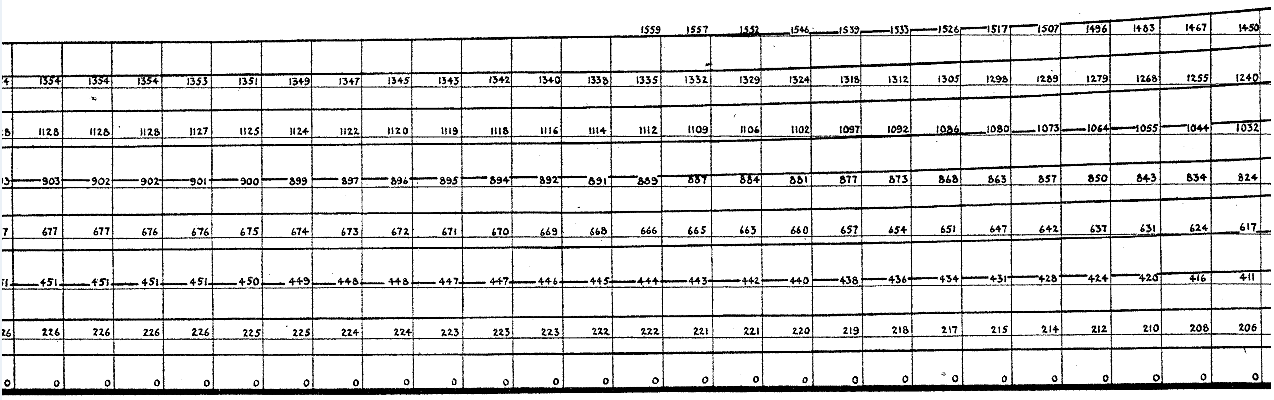
42. In both of Examples 7 and 8, the stream having passed the rigid barrier falls steadily to the lower of the two levels which are associated, according to § 10, with any particular rate of flow. We had expected to find solutions characterized by standing waves on the downstream side, but none in fact was found. In view of solutions obtained by Rayleigh (1883) and others, and which have been reviewed by Lamb (1932, §§ 242–5), this result calls for some enquiry.

Lamb (*loc. cit.*) remarks that in the absence of dissipative forces the problem is to some extent indefinite, in that one can always superpose an endless train of stationary waves having the wave-length appropriate to the stream-velocity; and that when the stream-depth h is finite there will be indeterminateness or not, according as the stream velocity c is less or greater than $\sqrt{(gh)}$. Rayleigh, treating a stream of infinite depth, avoided the indeterminateness by postulating dissipative forces of a somewhat arbitrary kind. Lamb (§ 245) discards the frictional terms in his analysis, but ultimately adjusts his solution (by introducing a uniform train of waves) so as to make the surface deformation insensible far upstream of the disturbance. *His treatment is restricted by his assumption that the elevation of the free surface is everywhere infinitesimal* (otherwise solutions could not be superposed): this means that in his problem the mean depth is the same on both sides of the disturbance, as was not the fact in our Examples 7 and 8. His results suggest that wave-motion, when it occurs downstream by reason of the disturbance caused by the barrier, may also extend upstream if (as in this paper) no dissipative forces are assumed. (If waves are formed downstream, they must (far from the barrier) have a wave-length compatible with the stream-depth (h, § 35). But then there is no reason why waves should not also extend upstream, similarly appropriate as regards wave-length.)

Allen's theorem (§ 12) shows that the free stream-line cannot fall below the lower of the two 'asymptotic levels', and Bernoulli's theorem shows that it cannot rise to stagnation level, unless at a cusped wave-crest (cf. § 37). If then waves can occur downstream, the free surface must fluctuate about the upper asymptotic level, and it must have restricted amplitude, so the disturbance caused by the barrier cannot be large. We have tried by extending the argument of § 12 to establish either the possibility or impossibility of wave-motion confined to the downstream side; but the results were inconclusive.

* Both quantities have been multiplied by $100\sqrt{12}$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES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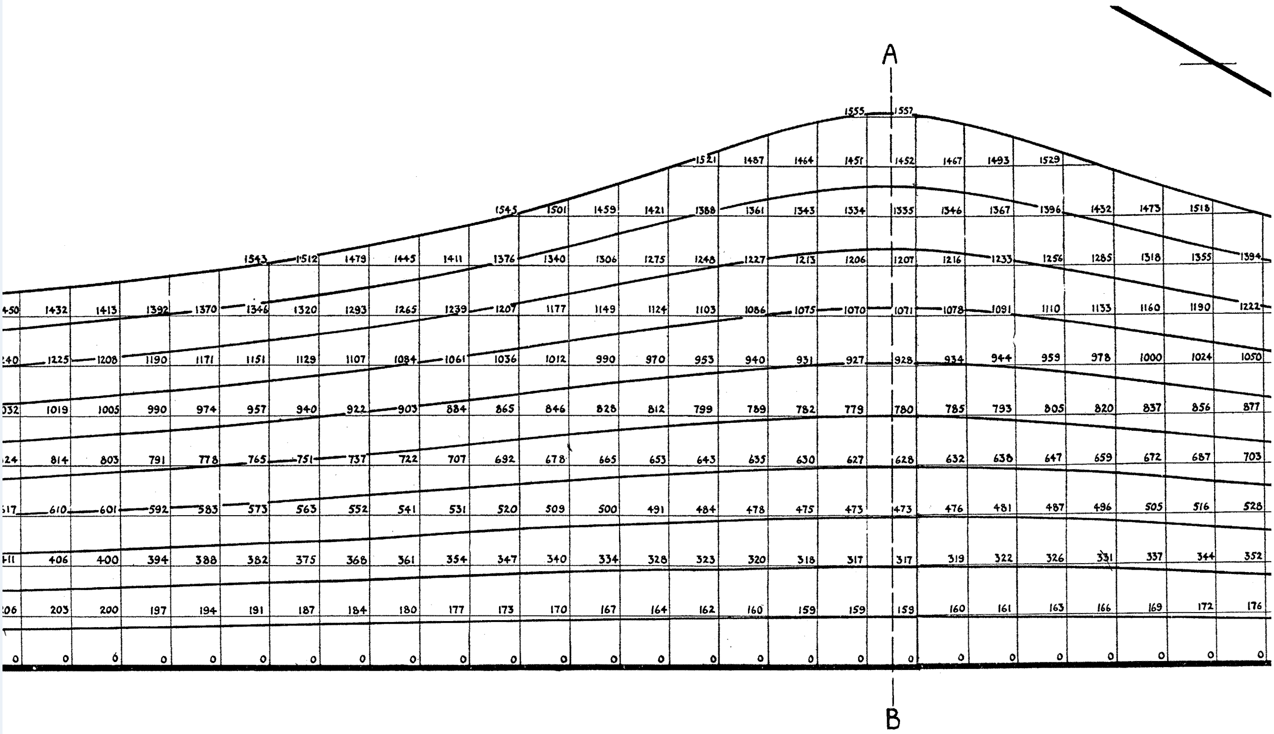
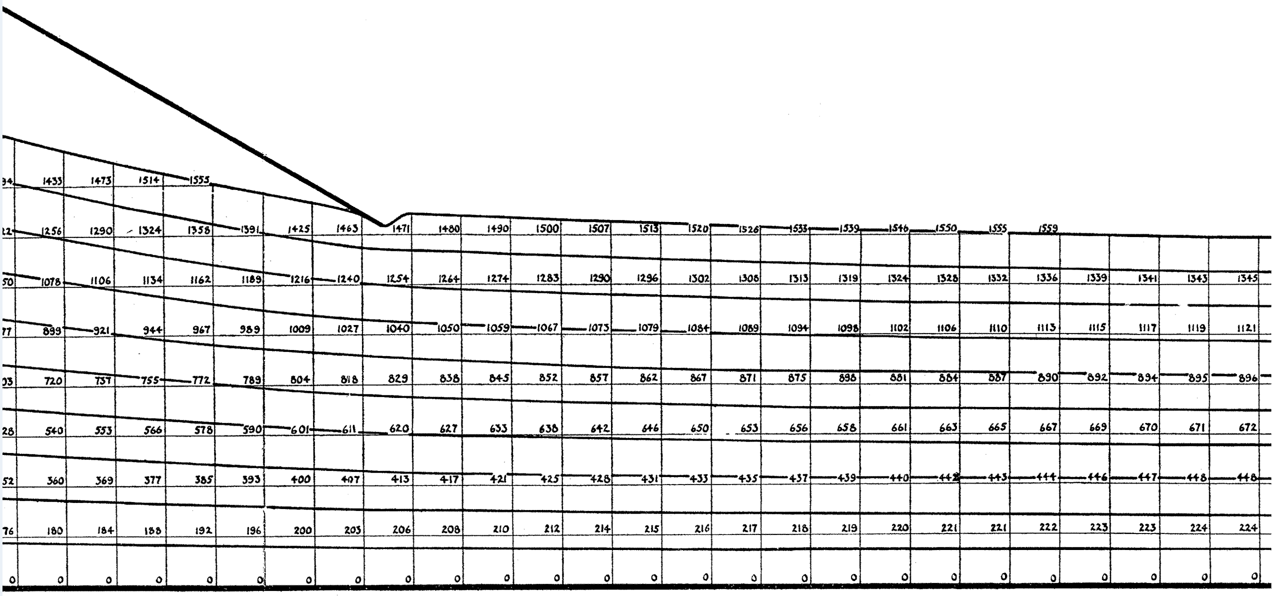


FIGURE 27

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Phi



34 1433 1473 1514 1553
12 1256 1290 1324 1358 1391 1425 1463 1471 1480 1490 1500 1507 1513 1520 1526 1533 1539 1546 1550 1555 1559
50 1078 1106 1134 1162 1189 1216 1240 1254 1264 1274 1283 1290 1296 1302 1308 1313 1319 1324 1328 1332 1336 1339 1341 1343 1345
77 899 921 944 967 989 1009 1027 1040 1050 1059 1067 1073 1079 1084 1089 1094 1098 1102 1106 1110 1113 1115 1117 1119 1121
03 720 737 755 772 789 804 818 829 838 845 852 857 862 867 871 875 898 881 884 887 890 892 894 895 896
28 540 553 566 578 590 601 611 620 627 633 638 642 646 650 653 656 658 661 663 665 667 669 670 671 672
52 360 369 377 385 393 400 407 413 417 421 425 428 431 433 435 437 439 440 442 443 444 446 447 448 448
76 180 184 188 192 196 200 203 206 208 210 212 214 215 216 217 218 219 220 221 221 222 223 223 224 224
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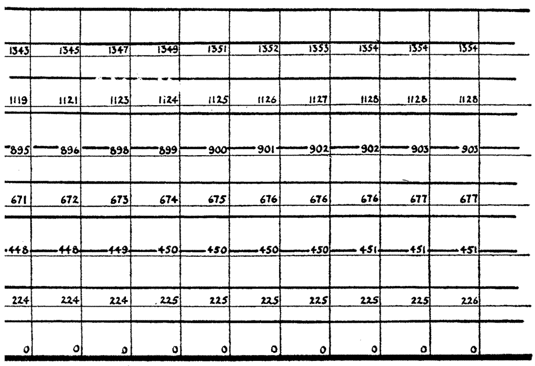

(*Facing p.* 153)

Example 9. *Scott Russell's 'solitary wave'*

43. The question cannot be settled by computation, because our boundary conditions admit the possibility of wave-motion in the oncoming stream, and until a computation is far advanced we cannot say whether it is tending to a solution in which the up-stream flow is undisturbed. An example of the sort of thing that happens is given in figure 27—a solution obtained in our study of the planing surface (§§ 40–1): we had expected a train of waves confined to the downstream side; but the computations 'took charge', and in the event we were left with a small highly localized disturbance by the planing surface of a wave extending to infinity in both directions.

That the disturbance is negligible is shown by the almost exact equality of the ψ-values on lines just to right and left of the vertical AB in figure 27: AB accordingly is (within the accuracy of our computations) a line of symmetry about which the left-hand side may be reflected. But then the planing surface disappears, leaving a symmetrical wave which extends to infinity. This is recognizable as an example—obtained fortuitously—of Scott Russell's 'solitary wave' (Lamb 1932, § 252); that is, a wave consisting of a single elevation which—as Russell found in experiment (1844)—can travel for a considerable distance along a uniform canal, with little or no change. (Waves of depression he found to break up into series of shorter waves.)

In its extreme form the wave must have a sharp crest of 120° angle, and McCowan (1894) showed that the maximum elevation will then approximate to 0·78 of the undisturbed stream depth h, the wave-velocity (c, § 34) to $\sqrt{(1\cdot 56gh)}$; i.e., in our notation (§ 35), h will approximate to $2/3\cdot 56 = 0\cdot 562$. Clearly the conditions in figure 27 approximate to this extreme case, and in it 0·78 is replaced by 0·60, 0·562 by 0·576.

Section III. Free stream-lines forming boundaries of 'wakes'

44. Finally we exemplify a class of problem confronted in aeronautics and in naval architecture—namely, the determination of 'wakes' formed in rear of bodies moving through fluid initially at rest. When the moving body has sharp edges, it is easy to decide the point from which the free stream-line will spring; but when the body has a curved profile, the free stream-line may start from any point either on its upstream or downstream side, and then the classical method (§ 1) is no longer adequate. A few transformations appropriate to curved boundaries have been devised, but we are not aware of any solution involving a restricted wake, like those numbered 60 and 90 in figure 28. Experiment, on the other hand, suggests that wakes of this kind can occur,—e.g. by entrainment of air in rear of a sphere projected into water.

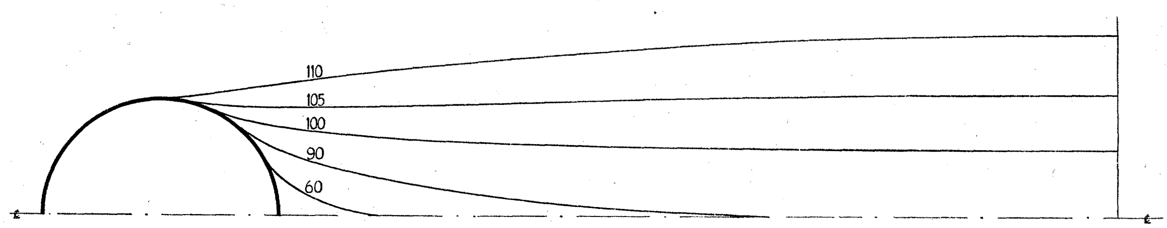

Figure 28

This paper, being concerned with computation, is not the place for a discussion of the way in which air can be thus entrained. Assuming entrainment to occur, we have sought to determine the resulting wake for a range of air pressures.

45. Gravity being without importance, the classical method can be applied when the flow is laminar and the moving boundary rectilinear, and it indicates that in some circumstances the wake may extend to infinity. But numerical computation must be restricted to a field of finite extent, so it is necessary to consider what boundary conditions should be imposed in applying Relaxation Methods, whether to laminar or to axially-symmetrical problems. We may not postulate that the 'undisturbed stream-function'

$$\psi = \lambda y \quad \text{or} \quad \psi = \lambda r^2 \quad (\lambda \text{ const.}) \tag{50}$$

will represent the motion of the fluid at all points distant from the moving body, for that would be to restrict the disturbance. It would seem advisable, after restricting the field of computation to a large rectangle, to require the pressure to be atmospheric at all points on its boundary.

In laminar problems, this will mean that everywhere on the rectangle we must satisfy

$$q^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 = \lambda^2,$$

or the equivalent conditions

$$\frac{\partial\psi}{\partial x} = -\lambda \sin\theta, \qquad \frac{\partial\psi}{\partial y} = \lambda\cos\theta, \tag{51}$$

in which θ denotes the local inclination of a stream-line (ψ = const.) to the axis of x. It is to be expected that θ will be small on the rectangular boundary, so as a starting assumption we may make it zero and replace (51) by

$$\left.\begin{aligned} \frac{\partial\psi}{\partial y} &= \lambda \text{ on the sides parallel to } Ox, \\ \frac{\partial\psi}{\partial x} &= 0 \text{ on the sides parallel to } Oy; \end{aligned}\right\} \tag{52}$$

then, having found a first approximation to the wanted stream-function, we can (if necessary) make a closer estimate of θ along the boundary, substitute that estimate in (51), and proceed to a second approximation. Axially-symmetrical problems (§§ 49–50) can be treated similarly.

Example 10. *The wake behind a circular cylinder*

46. Our first example is a case of laminar flow, being concerned with a wake of constant pressure behind a circular cylinder. In Rayleigh's treatment (1876*b*) of a lamina presented broadside to the oncoming stream, the pressure in the wake was made equal to the 'pressure at infinity'. Here, assuming only that it is *uniform* (as it must be, very nearly, when the wake consists of air), we compute free stream-lines for a range of specified 'wake pressures'.

The wanted free stream-line will be part of the stream-line $\psi = 0$, the other parts being (i) the line of symmetry ahead of the cylinder, (ii) that part of the rigid surface (starting from the 'stagnation point') with which the fluid remains in contact. The pressure need not be

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

uniform over (i) or (ii), but we shall postulate that it has no discontinuity at the point where the stream-line becomes free. Then (when the form of the cylinder is specified) the computational problem is defined.

47. It will be discussed most easily from the standpoint of Method (B), § 13. Given the pressure in the wake, we have q and therefore $\partial\psi/\partial\nu$ for all points on the free stream-line, and can interpret $\partial\psi/\partial\nu$ as edge-loading on a membrane. At points remote from the cylinder (*when the wake is closed*) the velocity of the fluid will have uniform magnitude and direction, so the stream-function (i.e. the displacement of the membrane) is known. We have now to make allowance for the cylinder and for the disturbance due to edge-loading on that part of the stream-line which is 'free'.

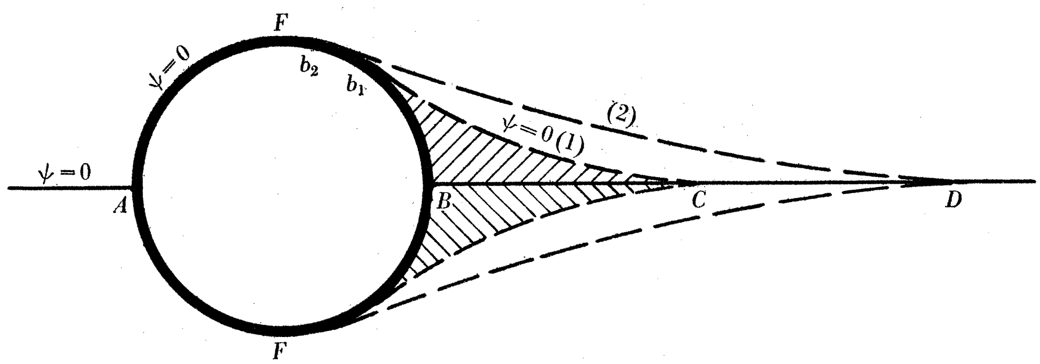

FIGURE 29

Suppose that the stream-function (ψ_0, say) has been determined for irrotational flow that does *not* involve the formation of a wake behind the cylinder, and that the specified 'wake pressure' is higher than the pressure at infinity. Then the gradient ($\mathbf{q}$, say) of the wanted stream-function ψ must have, on the boundary of the wake, a value lower than the gradient (q_0, say) of ψ_0 at points remote from the cylinder; but q_0 will be less than $\mathbf{q}$ at the rear stagnation-point B and over some known length BC of the line of symmetry (figure 29), since it increases steadily from zero at B to its limiting value far downstream. In the membrane analogue, imagine that the membrane is slit along BC, and that a downward edge-loading representative of the wanted gradient $\mathbf{q}$ is applied to its edge BC: then clearly (from what has been said above) this loading will depress the shaded part of the membrane so as to move the contour ($\psi = 0$) to some such position as is shown by the broken line marked (1). On the other side of BC the membrane will be lifted, so that (when the plane-potential problem has been solved) a solution follows in which a finite wake (shaded) adheres to the cylinder on its downstream side.

This first solution (ψ_1, say) will not be exact, because the membrane will not have the wanted gradient ($\mathbf{q}$) at points on the line b_1C. But imposing the wanted gradient along b_1C and along that part CD (if any) of the line of symmetry in which $\partial\psi_1/\partial\nu < \mathbf{q}$, we can proceed to a second solution indicated by the line marked (2) in figure 29; and by repetition of this process (if it converges) we shall arrive, eventually, at a solution satisfying all of the imposed conditions. Complete liquidation of residuals will not be necessary except in the final stage of the computation; for *provided that all residuals are negative, the stream-line can safely be moved as far as our computations would suggest.*

From the membrane analogue it is clear that

(1) the boundary of the wake will have a horizontal tangent (i.e. a cusp) where it joins the line of symmetry;*

(2) the specified gradient $\mathbf{q}$ may not exceed some limiting value; for otherwise the edge-loading will eventually carry the broken-line boundaries past the points (F, F, figure 29) at which q_0 has its largest value, and then no solution will be obtained. It appears (cf. figure 28) that in fact the limiting 'free' boundary leaves the cylinder at F, F;

(3) the wake will extend to infinity when the 'wake pressure' is atmospheric. For then the gradient q_0 is correct far downstream, so that there the imposed edge-loading will be sustained: elsewhere q_0 will be too small, so the edge loading will depress the whole line of symmetry.

48. Method (B) was actually used in this manner to start the computations for some of the cases summarized in figure 28, but all were completed by Method (A). The number attached to each wake boundary in figure 28 gives the relevant value of $\mathbf{q}$ expressed as a percentage of the velocity at infinity. The wake is restricted when this percentage is less than 100, but extends to infinity when it is 100 or over; so prediction (3) of § 47 is confirmed.

Figure 30 records the accepted solution for percentage 100, the actual gradient at the wake boundary having values which are shown for some points and of which the error nowhere exceeded 0·8 %. (Numerals lying outside the wake give values of the stream-function ψ.) The computations were generally similar to those entailed by preceding examples, the nets being graded in the neighbourhood of the cylinder and boundary. In every case of a wake extending to infinity, the stream-lines at the rectangular boundary were so nearly parallel as to render unnecessary the process of continued approximation which is sketched in § 45.

It may be said with confidence that the same procedure could have been applied successfully to any shape of cylinder.

Example 11. *The wake behind a sphere*

49. Wakes having axial symmetry are much harder to compute. The governing equation is (10) of § 5, and the velocity at (r, z) is

$$q = \frac{1}{r}\frac{\partial \psi}{\partial \nu},$$

ν denoting the normal to the stream-line; so the double boundary condition to be satisfied on the free stream-line is

$$\psi = 0, \quad \frac{1}{r}\frac{\partial \psi}{\partial \nu} = \mathbf{q} \quad (\mathbf{q} \text{ specified}). \tag{53}$$

But on the axis of symmetry both r and $\partial\psi/\partial\nu$ tend to zero (ψ as defined in § 5 having the form λr^2). Therefore at a cusp—which we must expect when the specified wake pressure is higher than the pressure at infinity—the quantity to be evaluated in computation assumes the indeterminate form 0/0.

* That the wake must have a cusp at the point of junction is also evident from hydrodynamical considerations. If its two sides met at an angle, the velocity would be zero at the point of junction, whereas it must have there, as elsewhere, the specified value $\mathbf{q}$.

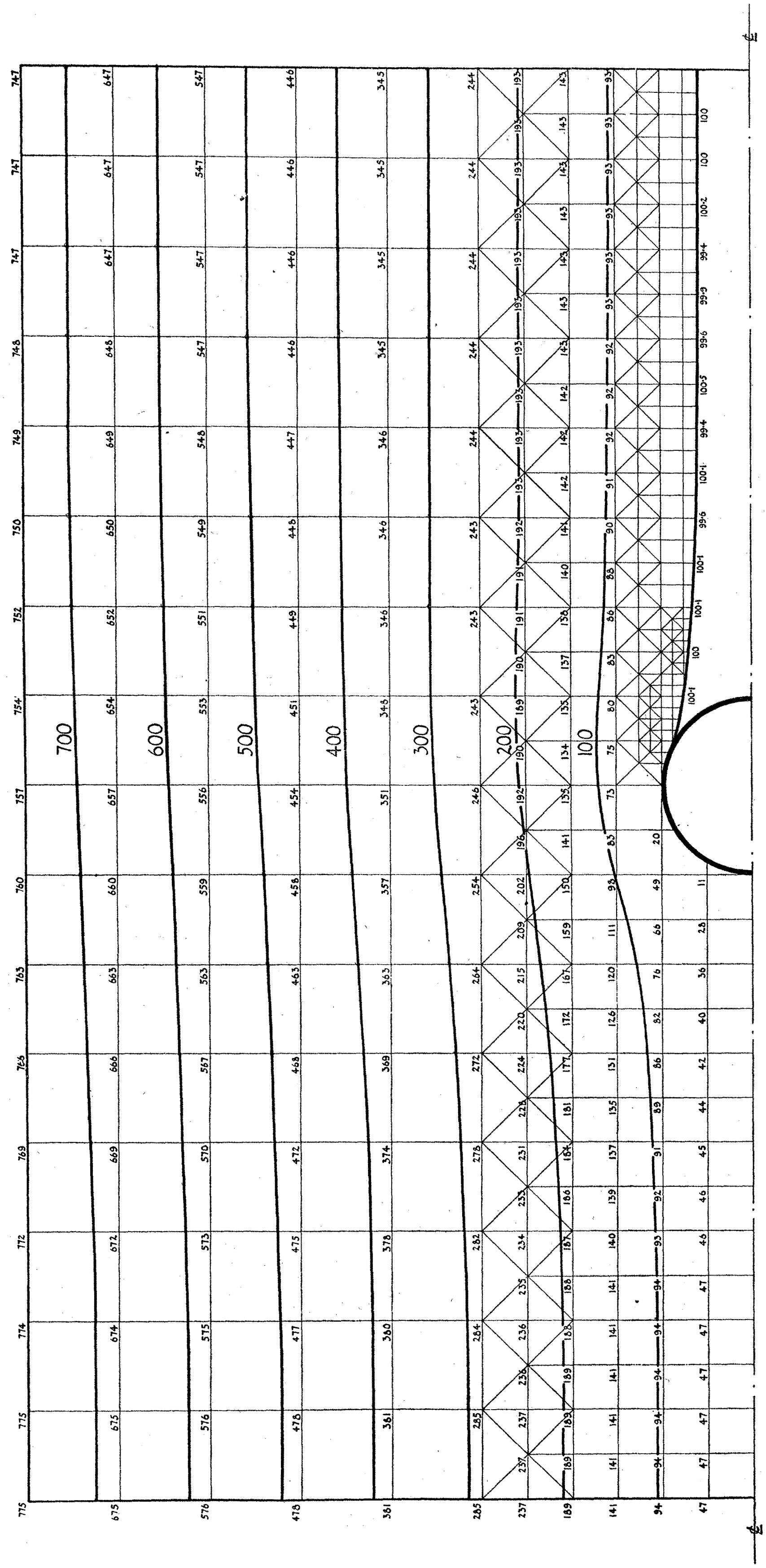

FIGURE 30

Thus a change of variables is necessitated. After some abortive attempts, we adopted the transformation

$$r = 2\sqrt{R}, \tag{54}$$

which alters the governing equation (10) to

$$\frac{\partial^2\psi}{\partial R^2}+\frac{1}{R}\frac{\partial^2\psi}{\partial z^2}=0, \tag{55}$$

and the equation of the spherical boundary ($r^2+z^2=1$) to

$$4R+z^2=1. \tag{56}$$

Thereby we obtained, in place of (53), the boundary conditions

$$\psi=0, \quad \frac{1}{2}\frac{\partial\psi}{\partial R}=\mathbf{q}\cos(r,\nu) \quad (\mathbf{q}\text{ specified}), \tag{57}$$

(r,ν) denoting the slope of the free stream-line in the r-z plane. Then, *keeping record of the changing boundary in the r-z plane*, we could decide on the necessary adjustments and thus be left (at every stage in the computation) with a definite problem for solution *in the plane of z and R.*

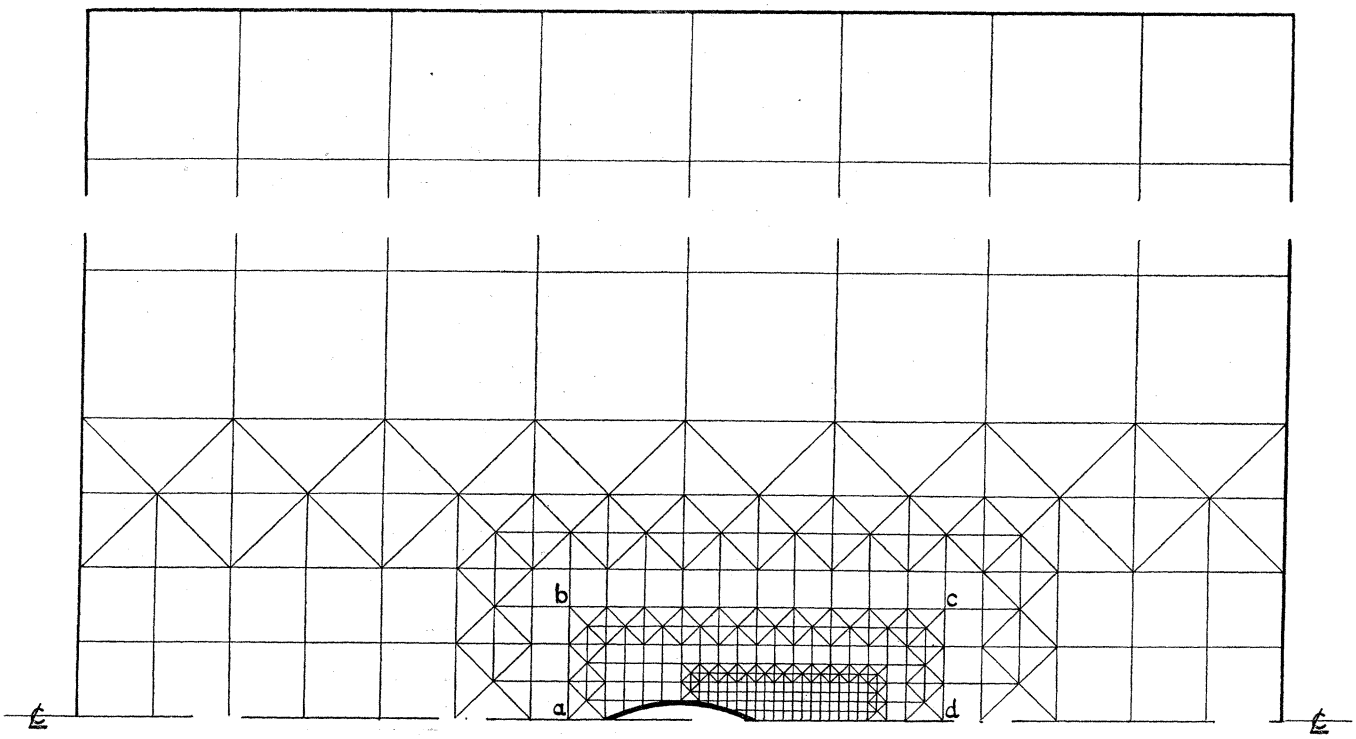

FIGURE 31

50. Some features of this computational problem are shown in figure 31. The transformation (54) yields a rigid boundary of finer lines than the original sphere, and makes the field of computation deeper than before in relation to its breadth; it also entails a use of grading that starts (apparently) at a greater distance from the rigid boundary. Within a region *abcd* immediately adjacent to the boundary the mesh-side *a* is very small.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

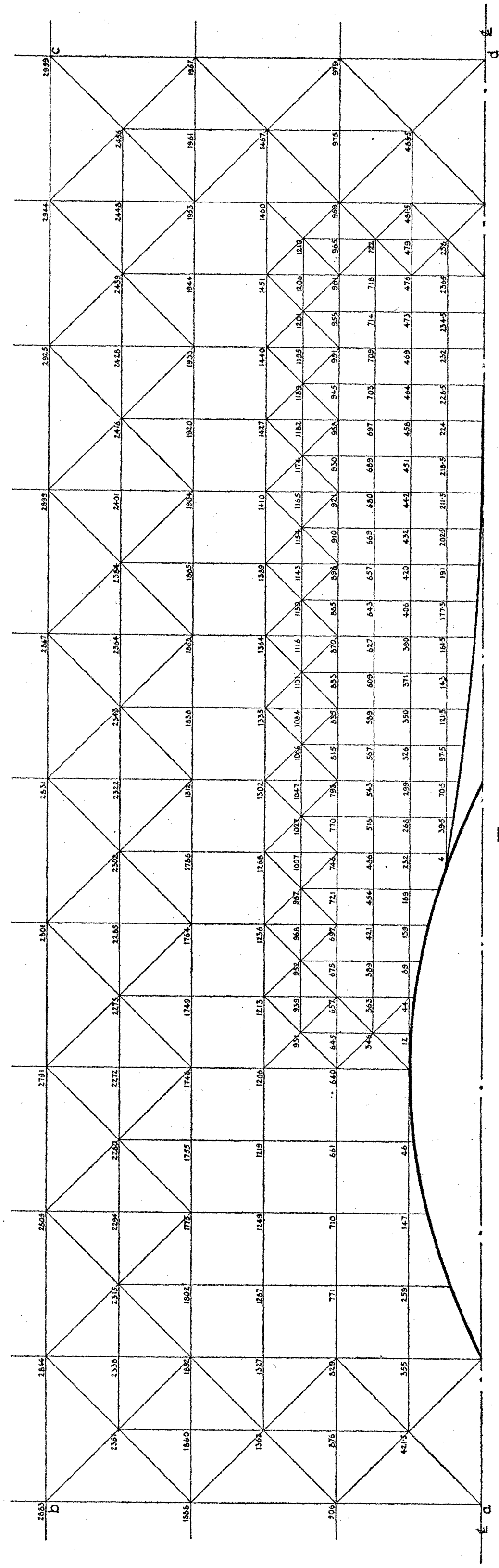

FIGURE 32

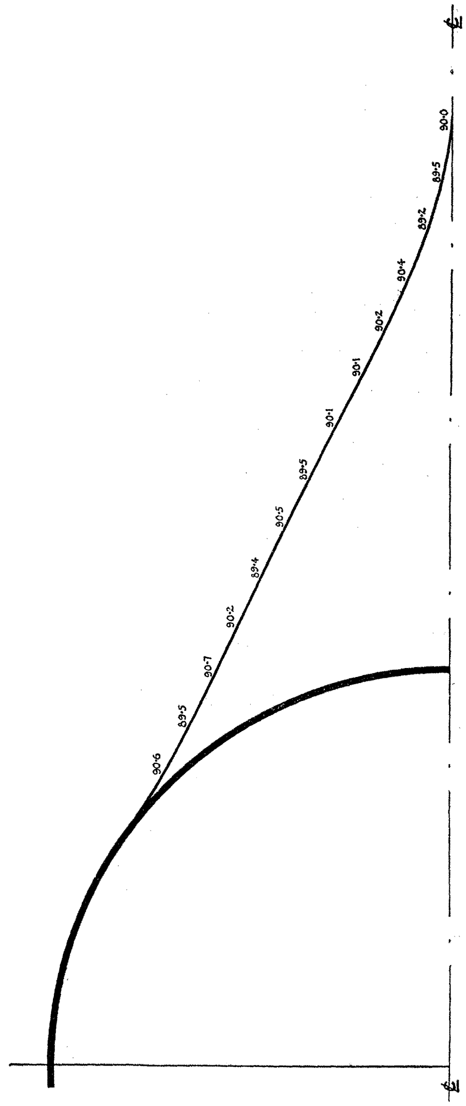

FIGURE 33

Figure 32 records the accepted solution inside *abcd* for the case where $\mathbf{q}$, in (53), is 90 % of the velocity at infinity: the numerals are values of the stream-function ψ. Figure 33 shows the form of the computed wake and (by numerals just outside the wake boundary) the accuracy to which the required velocity is there attained. The errors nowhere exceed 0·9 %.

Conclusion

51. Regarded as tasks for a computer, problems concerned with 'free' stream-lines are among the hardest yet attempted in this series. Only the tentative quality of Relaxation Methods permits (as in Part VII) a double boundary condition to be satisfied on a boundary initially unknown; and here an essential instability (non-existent in Part VII) makes any tentative solution liable to diverge. Moreover, abnormally fine nets are necessitated by the rapid curvature of free stream-lines at points where they first spring clear of rigid boundaries.* 'Graded nets' have proved indispensable.

It is not claimed that our techniques can be applied with confidence by an inexperienced computer; but we may, on the other hand, claim that in a wide variety of problems they have led to solutions which (within the accuracy permitted by our finite-difference approximations) closely satisfy all of the imposed conditions. That these have entailed much labour is hardly an objection in view of their scientific interest, seeing that few of them could (as yet) have been obtained by other methods; and a return for the labour is given in the detail that they provide. Given an analytical solution, it is a laborious matter, usually, to trace the form of the free stream-line. Our treatment, *which is applicable to any shape of rigid boundary*, gives the forms both of this and of other stream-lines throughout the whole field of computation.

Grateful acknowledgement is made to Mr D. N. de G. Allen, for advice and help in this investigation; also to the Advisory Council of the Department of Scientific and Industrial Research, for grants which have enabled one of us (G.V.) to pursue this and other investigations during the past four years.

* At such a point the free stream-line may have infinite curvature, although its direction changes continuously. Thus in the exact solution cited in the footnote to § 22,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \tan\theta \to 0 \text{ as } \theta \to 0,$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{d\theta}\Big/\frac{dx}{d\theta} = \frac{2\pi}{b-c}\cot\frac{\theta}{2}\sec^3\theta \to \infty \text{ as } \theta \to 0.$$

This means that no finite mesh-size will give exact results, and the computer must decide what amount of 'definition' is desirable.

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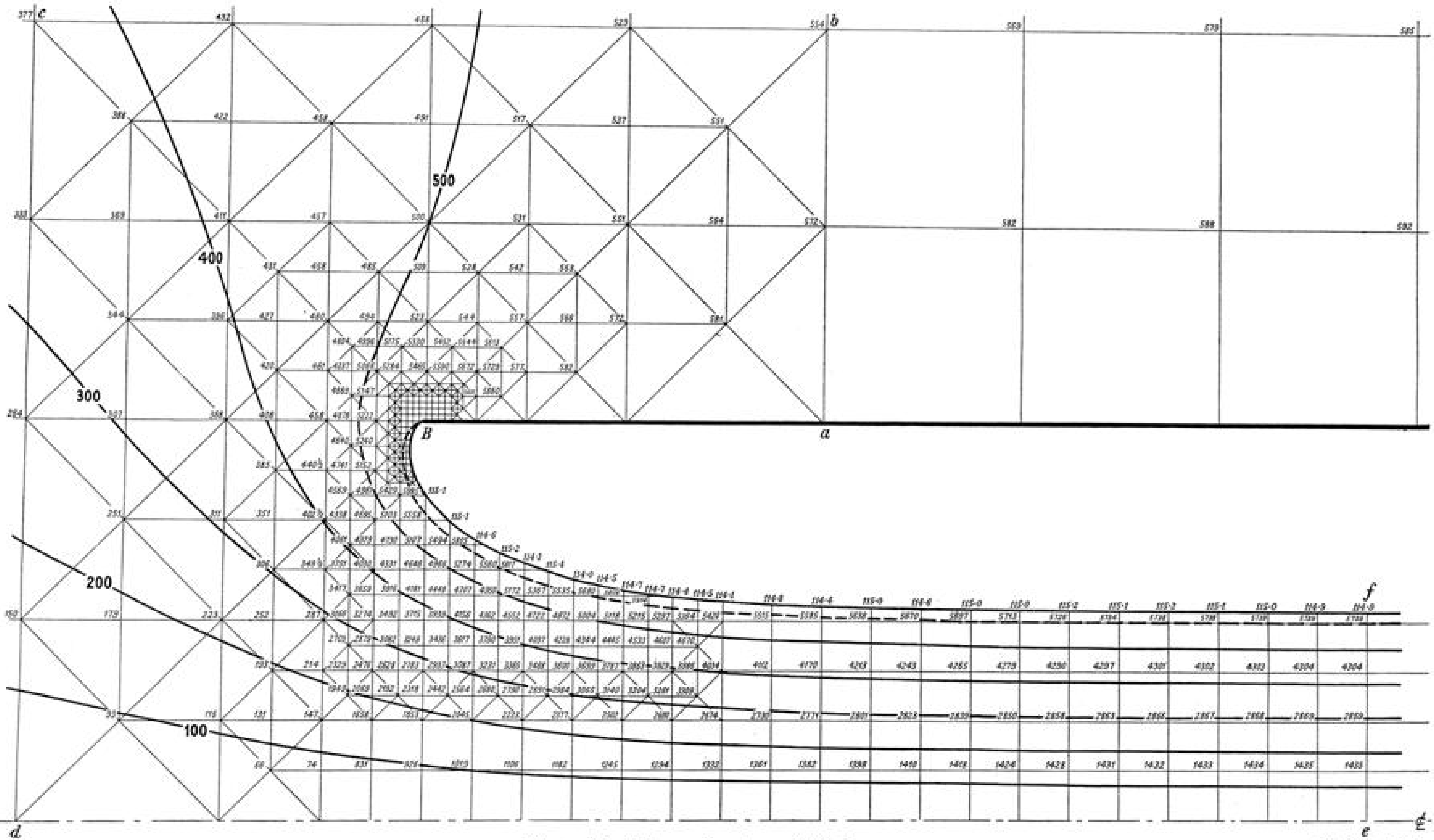

FIGURE 11. Values and contours of 100ψ.

(Facing p. 132)

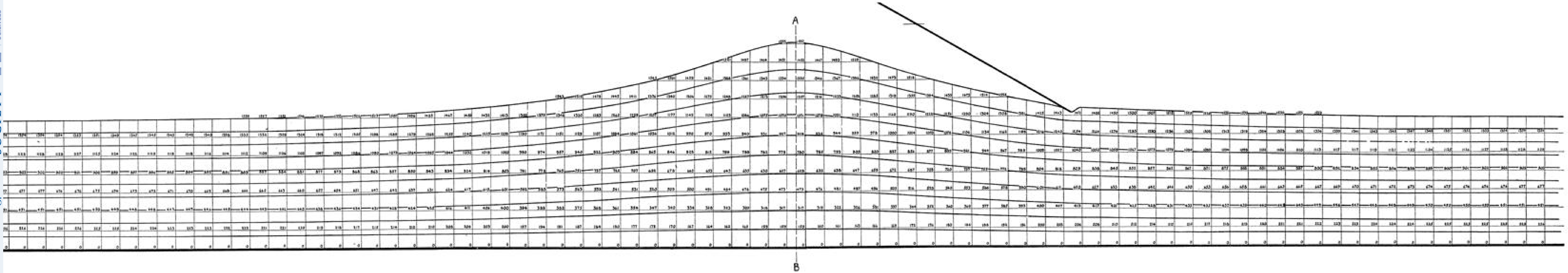

Figure 27

(*Facing p.* 153)